MINISTRY OF EDUCATION
DIPLOMA IN
INFORMATION COMMUNICATION TECHNOLOGY

KENYA INSTITUTE OF CURRICULUM DEVELOPMENT
STUDY NOTES

Computational Mathematics

MODULE I: SUBJECT NO 5

Contents

CHAPTER 1: INTRODUCTION TO COMPUTATIONAL MATHEMATICS

Equations

In mathematics an **equation** is an expression of the shape $A = B$, where A and B are expressions containing one or several variables called *unknowns*.

An algebraic equation or polynomial equation is an equation in which both sides are polynomials

A **system of polynomial equations** is a set of simultaneous equations $f_1 = 0, ..., f_h = 0$ where the f_i are polynomials in several variables, say $x_1, ..., x_n$, over some field k.

Algebraic equation

Mathematical statement of equality between algebraic expressions. An expression is algebraic if it involves a finite combination of numbers and variables and algebraic operations (addition, subtraction, multiplication, division, raising to a power, and extracting a root). Two important types of such equations are linear equations, in the form $y = ax + b$, and quadratic equations, in the form $y = ax^2 + bx + c$. A solution is a numerical value that makes the equation a true statement when substituted for a variable. In some cases it may be found using a formula; in others the equation may be rewritten in simpler form.

1. **A linear equation** is an algebraic equation in which each term is either a constant or the product of a constant and (the first power of) a single variable.

A common form of a linear equation in the two variables x and y is

$$y = mx + b,$$

where m and b designate constants (parameters). The origin of the name "linear" comes from the fact that the set of solutions of such an equation forms a straight line in the plane. In this particular equation, the constant m determines the slope or gradient of that line, and the constant term b determines the point at which the line crosses the y-axis, otherwise known as the y-intercept.

General (or standard) form

In the general (or standard) form the linear equation is written as:

$$Ax + By = C,$$

where A and B are not both equal to zero. The equation is usually written so that $A \geq 0$, by convention. The graph of the equation is a straight line, and every straight line can be represented by an equation in the above form. If A is nonzero, then the x-intercept, that is, the x-coordinate of the point where the graph crosses the x-axis (where, y is zero), is C/A. If B is nonzero, then the y-intercept, that is the y-

coordinate of the point where the graph crosses the *y*-axis (where x is zero), is *C*/*B*, and the slope of the line is −*A*/*B*. The general form is sometimes written as:

$$ax + by + c = 0,$$

where *a* and *b* are not both equal to zero. The two versions can be converted from one to the other by moving the constant term to the other side of the equal sign.

Slope–intercept form

$$y = mx + b,$$

where *m* is the slope of the line and *b* is the *y*-intercept, which is the *y*-coordinate of the location where line crosses the *y* axis. This can be seen by letting $x = 0$, which immediately gives $y = b$. It may be helpful to think about this in terms of $y = b + mx$; where the line passes through the point (0, *b*) and extends to the left and right at a slope of *m*. Vertical lines, having undefined slope, cannot be represented by this form.

Point–slope form

$$y - y_1 = m(x - x_1),$$

where *m* is the slope of the line and (x_1,y_1) is any point on the line.

The point-slope form expresses the fact that the difference in the *y* coordinate between two points on a line (that is, $y - y_1$) is proportional to the difference in the *x* coordinate (that is, $x - x_1$). The proportionality constant is *m* (the slope of the line).

Two-point form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

where (x_1, y_1) and (x_2, y_2) are two points on the line with $x_2 \neq x_1$. This is equivalent to the point-slope form above, where the slope is explicitly given as $(y_2 - y_1)/(x_2 - x_1)$.

Multiplying both sides of this equation by $(x_2 - x_1)$ yields a form of the line generally referred to as the **symmetric form**:

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1).$$

Intercept form

$$\frac{x}{a} + \frac{y}{b} = 1,$$

Where *a* and *b* must be nonzero. The graph of the equation has *x*-intercept *a* and *y*-intercept *b*. The intercept form is in standard form with *A*/*C* = 1/*a* and *B*/*C* = 1/*b*. Lines that pass through the origin or which are horizontal or vertical violate the nonzero condition on *a* or *b* and cannot be represented in this form.

Matrix form

Using the order of the standard form

$$Ax + By = C,$$

one can rewrite the equation in matrix form:

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C \end{pmatrix}.$$

Further, this representation extends to systems of linear equations.

$$A_1x + B_1y = C_1,$$

$$A_2x + B_2y = C_2,$$

becomes

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Since this extends easily to higher dimensions, it is a common representation in linear algebra, and in computer programming. There are named methods for solving system of linear equations, like Gauss-Jordan which can be expressed as matrix elementary row operations.

2. **A quadratic equation** is a univariate polynomial equation of the second degree. A general quadratic equation can be written in the form

$$ax^2 + bx + c = 0,$$

where *x* represents a variable or an unknown, and *a*, *b*, and *c* are constants with $a \neq 0$. (If $a = 0$, the equation is a linear equation.). The constants *a*, *b*, and *c* are called respectively, the quadratic coefficient, the linear coefficient and the constant term

A quadratic equation with real or complex coefficients has two solutions, called *roots*. These two solutions may or may not be distinct, and they may or may not be real.

Methods to Solve Quadratic Equations

Below are the four most commonly used methods to solve quadratic equations. Click on any link to learn more about any of the methods.

- The Quadratic Formula (Quadratic formula in depth)
- Factoring
- Completing the Square
- Graph

Method 1 – Formula

The solution of a quadratic equation is the value of x when you set the equation equal to zero i.e. When you solve the following general equation: $0 = ax^2 + bx + c$
Given a quadratic equation: $ax^2 + bx + c$
The quadratic formula below will solve the equation for zero
The quadratic formula is:

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Examples of the quadratic formula to solve an equation

- **Example 1**

  Quadratic Equation: $\mathbf{y = x^2 + 2x + 1}$

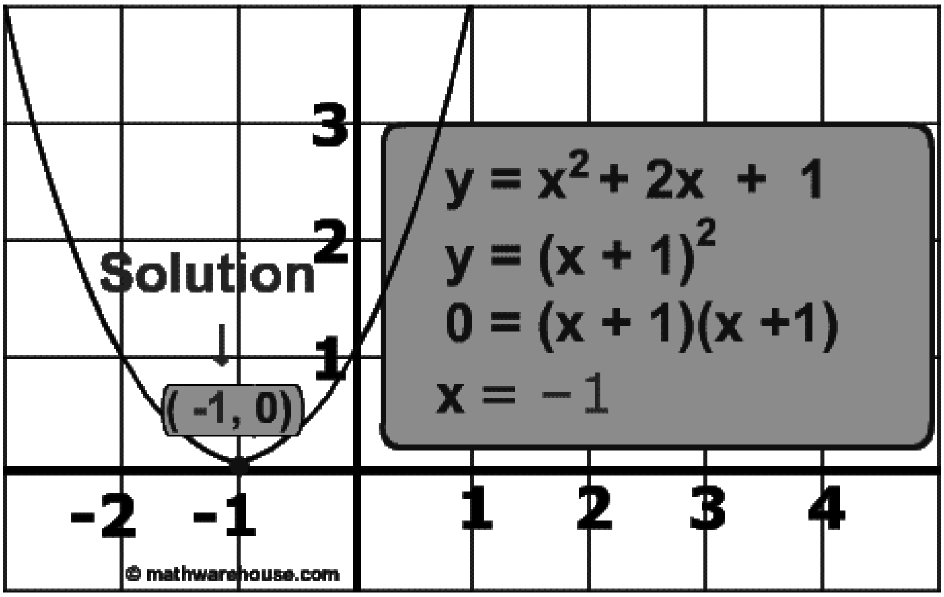
  - $a = 1$
  - $b = 2$
  - $c = 1$

Using the quadratic formula to solve this equation just substitutes a, b, and c into the general formula:

$$\frac{-2 \pm \sqrt{2^2-4(1)(1)}}{2(1)} \rightarrow \frac{-2 \pm \sqrt{4 - 4}}{2} \rightarrow$$

$$\rightarrow \frac{-2 \pm \sqrt{0}}{2} \rightarrow \frac{-2}{2} \rightarrow -1$$

Method 2 – Graph - Below is a picture representing the graph of $y = x^2 + 2x + 1$ and its solution



Method 3 – Factoring

The solution of a quadratic equation is the value of x when you set the equation equal to zero i.e. When you solve the following general equation: $0 = ax^2 + bx + c$

Given a quadratic equation: $ax^2 + bx + c$

One method to solve the equation for zero is to factor the equations.

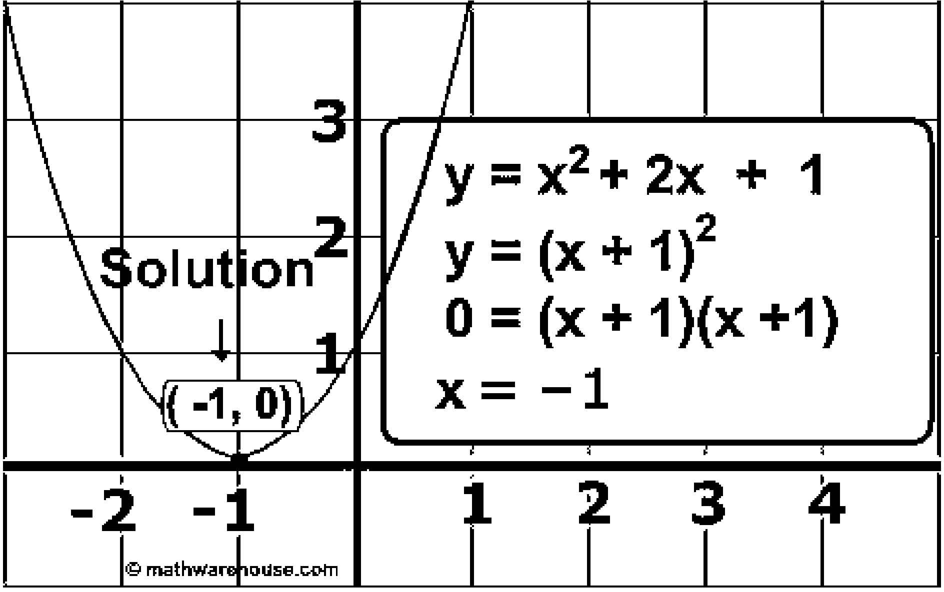
General Steps to solve by factoring

- Step 1) Create a factor chart for all factor pairs of **c**
    - A factor pair is just two numbers that multiply and give you **'c'**
- Step 2) Out of all of the factor pairs from step 1, look for the pair (if it exists) that add up to **b**

    - **Note:** if the pair does not exist, you must either complete the square or use the quadratic formula .
- Step 3) Insert the pair you found in step 2 into two binomials
- Step 4) Solve each binomial for zero to get the solutions of the quadratic equation.
- Example of how to solve a quadratic equation by factoring
- Quadratic Equation: **$y = x^2 + 2x + 1$**

<table>
<tr><th colspan="2">$y=ax^2+bx+c$
$y=x^2+2x+1$</th></tr>
<tr><td>1) Create a factor chart</td><td><table><tr><th>Factors of c (as pairs)</th><th>Sum of factors (must equal b)</th></tr><tr><td>1.1</td><td>1+1</td></tr></table></td></tr>
<tr><td>2) Determine which of factor pair of “c” has a sum of “b”</td><td>1,1</td></tr>
<tr><td>3) Insert that pair into binomial factors</td><td>y=(x+1)(x+1)</td></tr>
<tr><td>4) Solve each binomial for zero</td><td>0 = x + 1
-1 -1
-1 = x</td></tr>
</table>

Below is a picture representing the graph of y = x² + 2x + 1 as well as the solution we found by factoring



Method 4 – Completing the square

Formula for Completing the Square

First off, a little necessary vocabulary:
A perfect square trinomial is a polynomial that you get by squaring a binomial.(binomials are things like 'x + 3' or 'x − 5')
Examples of perfect square trinomials (the red trinomials)

- $(x+ 1)^2 = x^2 + 2x + 1$
- $(x+ 2)^2 = x^2 + 4x + 4$
- $(x+ 3)^2 = x^2 + 6x + 9$

Examples of trinomials that are **NOT** perfect square trinomials

- $(x+ 1)(x +2)= x^2 + 3x + 2$
- $(x+ 2)(x+5) = x^2 + 7x + 10$
- $(x+ 3)(x −3)= x^2 − 9$

To best understand the formula and logic behind completing the square, look at each example below and you should see the pattern that occurs whenever you square a binomial to produce a perfect square trinomial.

$(x+3)^2$	=	x^2+6x+9
	=	$x^2+2(3)x+3^2$
$(x+4)^2$	=	$x^2+8x+16$
	=	$x^2+2(4)x+4^2$
$(x+5)^2$	=	$x^2+10x+25$
	=	$x^2+2(5)x+5^2$
$(x+6)^2$	=	$x^2+12x+36$
	=	$x^2+2(6)x+6^2$
$(x+7)^2$	=	$x^2+14x+49$
	=	$x^2+2(7)x+7^2$
$(x+k)^2$	=	$x^2+2(k)x+k^2$

General Formula

↓

$$(x + \frac{t}{2})^2 = x^2 + 2(\frac{t}{2}) + (\frac{t}{2})^2$$

As the examples above show, the square of a binomial always follows the same pattern and formula.

Given a quadratic equation $\mathbf{x^2 + bx + c}$ that is a SQUARE OF A BINOMIAL

c is always the square of ½**(b)**

i.e. c = ½(b)²

3. **Simultaneous equations**

The terms ***simultaneous equations*** and *systems of equations* refer to conditions where two or more unknown variables are related to each other through an equal number of equations.

A set of two or more equations, each containing two or more variables whose values can simultaneously satisfy both or all the equations in the set, the number of variables being equal to or less than the number of equations in the set

There are Four methods to solve a system of equations:

1. Graphing
2. Substitution
3. Elimination or addition method
4. Matrices

For example

$6x + 3y = 12$

$5x + y = 7$

Method 1 – Graphing

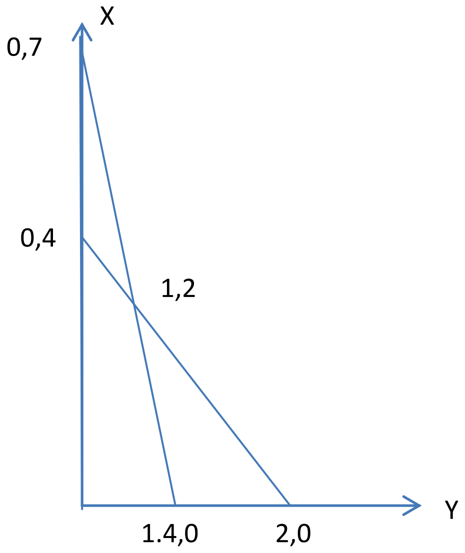
To solve by graphing, the two lines must be graphed on the same rectangular co-ordinate system. To graph each line, you need to find at least two points on each line. Set up an x and y charts on each. The points on which the two graphs intersect is the solution.

For $6x + 3y = 12$	
x	y
0	4
2	0

For $5x + y = 7$	
x	y
0	7
1.4	0

Select a value for one variable and solve for the other

\



Method 2 - Substitution

To solve by substitution, select one equation (it does not matter which of the equation) and solve for one of the variables (it does not matter which of the variables). Then substitute that expression into the other equation.

For $6x + 3y = 12$ I am going to solve for Y (why?, because I want to)

$6x + 3y = 12$
$(6x + 3y) -6x = 12 -6x$ therefore $3y=-6x+12$
$3y/3=-(6x+12)/3$ hence $y=-2x+4$

Since I solved for y, I am going to use this to substitute into the y variable of the second equation

For $5x + y = 7$
$5x + (-2x+4) = 7$ now solve for X
$5x-2x+4=7$ **=>**$3x+4=7$ **=>**$3x=3$ =>$x=1$

Now that we know what x is, use this to solve for y by using one of the original equations (it does no matter which of the equations)

I'll use 6x + 3y = 12(why?, because I want to)

6x + 3y = 12 **=>** 6(1) + 3y = 12 **=>** 6 + 3y = 12 **=>**3y = 12-6 **=>** 3y = 6
therefore y = 6/3=2

Therefore, our answer is (1, 2) which is our point of intersection.

Method 3 – Elimination or Addition method

The objective of elimination method is to add the equations such that like terms add up to zero. If no like terms add up to zero, the multiplication is used to alter the system of equations such that like terms do add up to zero.

If the equation

$$\begin{array}{l} 6x + 3y = 12 \\ 5x + y = 7 \\ \hline 11x+4y=19 \end{array}$$

were added now, no like terms world add up to zero

Therefore multiplication is required to alter the equation. To do this legally, remember that whatever you do on one side of the equation you must do the same on the other side of the equation.

To eliminate a variable, do the following

1. Decide which variable to eliminate(it does not matter which of the equation)

$$\begin{array}{l} 6x + 3y = 12 \\ 5x + y = 7 \end{array}$$

I'll pick the X term (why?, because I want to)

2. Looking at the variable from each equation, find the least common multiple of the coefficients (X and Y)

$$\begin{array}{l} 6x + 3y = 12 \\ 5x + y = 7 \end{array}$$

The least common multiple of 5 and 6 is 30

3. Now multiply each equation by the number which will make the co-efficient on the x variable 30

$$\begin{matrix} 5[6x + 3y = 12] & \longrightarrow & 30x + 15y = 60 \\ 6[5x + y = 7] & & 30x + 6y = 42 \end{matrix}$$

Notice that if this equations were added, the X terms would still not add up to zero, therefore one of the factors, 5 or 6, which was used to multiply with the equations must be negative, I'll chose 6 to be negative (why?, because I want to)

5[6x + 3y = 12]

-6[5x + y = 7]

→

30x + 15y = 60

-30x - 6y = -42

9y=18

Y=2

4. Now substitute this value into one of the original equations(it does not matter which of the equation) to solve for X
6x + 3y = 12(why? because I want to)

6x+3(2) =12 **=>** 6x+6=12 **=>** 6x=6 **=>** x=1

Therefore our answer is (1,2) which happens to be the point of intersection.

Method 4: Matrices

Example 1

EQUATIONS

- x + y + z = 6
- 2y + 5z = -4
- 2x + 5y - z = 27

A Matrix is an array of numbers, right?

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

Well, think about the equations:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 6 \\ & & 2y & + & 5z & = & -4 \\ 2x & + & 5y & - & z & = & 27 \end{array}$$

They could be turned into a table of numbers like this:

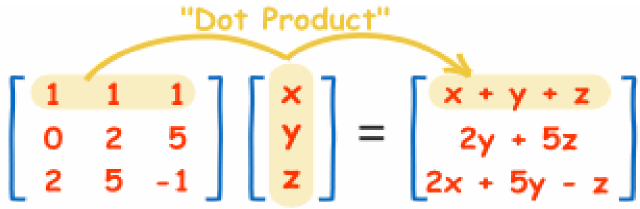
$$\begin{array}{ccccc} 1 & 1 & 1 & = & 6 \\ 0 & 2 & 5 & = & -4 \\ 2 & 5 & -1 & = & 27 \end{array}$$

We could even separate the numbers before and after the "=" into:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{array} \quad \text{and} \quad \begin{array}{c} 6 \\ -4 \\ 27 \end{array}$$

Now it looks like we have 2 Matrices.

In fact we have a third one, which is [x y z], and the way that matrices are ,we need to set it up like this:



$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y + z \\ 2y + 5z \\ 2x + 5y - z \end{bmatrix}$$

And we know that x + y + z = 6, etc, so we can write the system of equations like this:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

Pretty neat, hey?

The Matrix Solution

We can call the matrices "A", "X" and "B" and the equation becomes:

$$AX = B$$

Where A is the 3x3 matrix of x,y and z coefficients X is x, y and z, and B is 6, -4 and 27

Then (as shown on the **Inverse of a Matrix** page) the solution is this:

$$X = A^{-1}B$$

(Assuming we can calculate the Inverse Matrix A^{-1})

Then multiply A^{-1} by B (you can use the Matrix Calculator again):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-21} \begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix} = \frac{1}{-21} \begin{bmatrix} -105 \\ -63 \\ 42 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

And we are done! The solution is:

x = 5, y = 3 and z = -2

An introduction to MATRICES

Definition of Matrix

A matrix is a collection of numbers arranged into a fixed number of rows and columns. Usually the numbers are real numbers. In general, matrices can contain complex numbers but we won't see those here. Here is an example of a matrix with three rows and three columns:

$$\begin{array}{cc} & \text{Cols 1.......} \\ \text{Rows 1........} & \begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & 4.6 \\ 4 & -1 & 0 \end{pmatrix} \end{array}$$

The top row is row 1. The leftmost column is column 1. This matrix is a 3x3 matrix because it has three rows and three columns. In describing matrices, the format is:

Rows X columns

Each number that makes up a matrix is called an **element** of the matrix. The elements in a matrix have specific locations.

The upper left corner of the matrix is row 1 column 1. In the above matrix the element at row 1 col 1 is the value 1. The element at row 2 column 3 is the value 4.6.

Matrix Dimensions

The numbers of rows and columns of a matrix are called its **dimensions**. Here is a matrix with three rows and two columns:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3 \times 2}$$

Sometimes the dimensions are written off to the side of the matrix, as in the above matrix. But this is just a little reminder and not actually part of the matrix. Here is a matrix with different dimensions. It has two rows and three columns. This is a different "data type" than the previous matrix

$$\begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{0} & \mathbf{8} & \mathbf{4.6} \\ \mathbf{4} & \mathbf{-1} & \mathbf{0} \end{pmatrix} \quad \mathbf{2\times3}$$

Types of Matrices

Square matrix

If a matrix A has n rows and n columns then we say it's a square matrix.

In a square matrix the elements $a_{i,i}$, with i = 1,2,3,... , are called diagonal elements.
Remark. There is no difference between a 1 x 1 matrix and an ordinary number.

Diagonal matrix

A diagonal matrix is a square matrix with all de non-diagonal elements 0.

The diagonal matrix is completely defined by the diagonal elements.

Example.

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

`The matrix is denoted by diag(7 , 5 , 6)`

Row matrix

A matrix with one row is called a row matrix.

[2 5 -1 5]

Column matrix

A matrix with one column is called a column matrix.

```
[2]
[4]
[3]
[0]
```

Matrices of the same kind

Matrix A and B are of the same kind if and only if
A has as many rows as B and A has as many columns as B

```
[7  1  2]       [4  0  3]
[0  5  6]  and  [1  1  4]
[3  4  6]       [8  6  2]
```

The transposed matrix of a matrix

The n x m matrix B is the transposed matrix of the m x n matrix A if and only if
The ith row of A = the ith column of B for (i = 1,2,3,..m)
So $a_{i,j} = b_{j,i}$

The transposed matrix of A is denoted T(A) or A^T

```
          T
[7  1 ]       [7  0  3]
[0  5 ]    =  [1  5  4]
[3  4 ]
```

0-matrix

When all the elements of a matrix A are 0, we call A a 0-matrix.
We write shortly 0 for a 0-matrix.

An identity matrix I

An identity matrix I is a diagonal matrix with all the diagonal elements = 1.

```
[1]

[1 0]
[0 1]

[1 0 0]
[0 1 0]
[0 0 1]

. . .
```

A scalar matrix S

A scalar matrix S is a diagonal matrix whose diagonal elements all contain the same scalar value.
$a_{1,1} = a_{i,i}$ for (i = 1,2,3,..n)

```
[7  0  0]
[0  7  0]
[0  0  7]
```

The opposite matrix of a matrix

If we change the sign of all the elements of a matrix A, we have the opposite matrix -A.
If A' is the opposite of A then $a_{i,j}' = -a_{i,j}$, for all i and j.

A symmetric matrix

A square matrix is called symmetric if it is equal to its transpose.
Then $a_{i,j} = a_{j,i}$, for all i and j.

```
[7  1  5]
[1  3  0]
[5  0  7]
```

A skew-symmetric matrix

A square matrix is called skew-symmetric if it is equal to the opposite of its transpose.
Then $a_{i,j} = -a_{j,i}$, for all i and j.

```
[ 0  1 -5]
[-1  0  0]
```

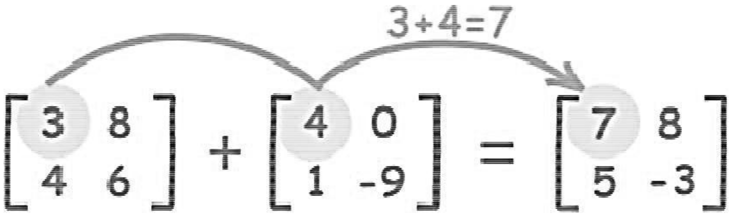
[5 0 0]

Operation of matrices

Addition of Matrices

Adding

To add two matrices: add the numbers in the matching positions:



These are the calculations:

3+4=7 8+0=8

4+1=5 6-9=-3

The two matrices must be the same size, i.e. the rows must match in size, and the columns must match in size.

Example: a matrix with **3 rows** and **5 columns** can be added to another matrix of **3 rows** and **5 columns**.

But it could not be added to a matrix with **3 rows** and **4 columns** (the columns don't match in size)

Denote the sum of two matrices A and B (of the same dimensions) by C = A + B.. The sum is defined by adding entries with the same indices

$$c_{ij} \equiv a_{ij} + b_{ij}$$

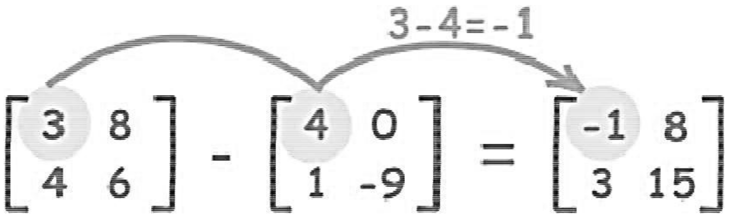
over all i and j.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Subtraction of Matrices

To subtract two matrices: subtract the numbers in the matching positions:



$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 3 & 15 \end{bmatrix}$$

These are the calculations:

3-4=-1   8-0=8

4-1=3 6-(-9)=15

Note: subtracting is actually defined as the ***addition*** *of a negative matrix: A + (-B)*

Subtraction is performed in analogous way.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

How to Multiply Matrices

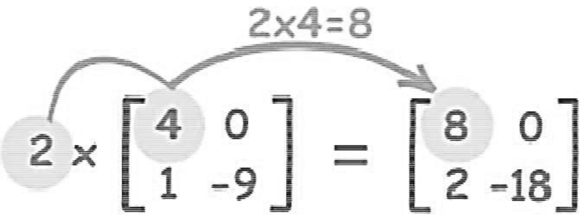
A Matrix is an array of numbers:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 3 Columns)

To multiply a matrix by a single number is easy:



$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

These are the calculations:

2×4=8 2×0=0 2×1=2

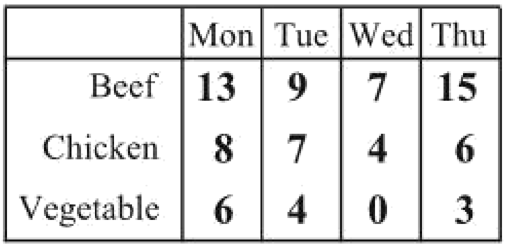
2×-9=-18

We call the number ("2" in this case) a **scalar**, so this is called "scalar multiplication".

Example: The local shop sells 3 types of pies.

- Beef pies cost **$3** each
- Chicken pies cost **$4** each
- Vegetable pies cost **$2** each

And this is how many they sold in 4 days:



	Mon	Tue	Wed	Thu
Beef	**13**	**9**	**7**	**15**
Chicken	**8**	**7**	**4**	**6**
Vegetable	**6**	**4**	**0**	**3**

Now think about this ... the **value of sales** for Monday is calculated this

way: Beef pie value + Chicken pie value + Vegetable pie value

\$3×13 + \$4×8 + \$2×6 = \$83

So it is, in fact, the "dot product" of prices and how many were sold:

$$(\$3, \$4, \$2) \bullet (13, 8, 6) = \$3\times 13 + \$4\times 8 + \$2\times 6 = \$83$$

We **match** the price to how many sold, **multiply** each, then **sum** the result.

In other words:

- The sales for Monday were: Beef pies: **\$3×13=\$39**, Chicken pies: **\$4×8=\$32**, and Vegetable pies: **\$2×6=\$12**. Together that is \$39 + \$32 + \$12 = **\$83**
- And for Tuesday: **\$3×9 + \$4×7 + \$2×4 = \$63**
- And for Wednesday: **\$3×7 + \$4×4 + \$2×0 = \$37**
- And for Thursday: **\$3×15 + \$4×6 + \$2×3 = \$75**

So it is important to match each price to each quantity.

Now you know why we use the "dot product".

And here is the full result in Matrix form:

$$\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix} \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \underset{\uparrow}{\$83} & \$63 & \$37 & \$75 \end{bmatrix}$$

$\$3 \times 13 + \$4 \times 8 + \$2 \times 6$

They sold **$83** worth of pies on Monday, **$63** on Tuesday, etc.

(You can put those values into the Matrix Calculator to see if they work.)

Rows and Columns

To show how many rows and columns a matrix has we often write

rows×columns. Example: This matrix is 2×3 (2 rows by 3 columns):

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

When we do multiplication:

- The number of **columns of the 1st matrix** must equal the number of **rows of the 2nd matrix**.
- And the result will have the same number of **rows as the 1st matrix**, and the same number of **columns as the 2nd matrix**.

Example:

$$\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix} \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \underset{\uparrow}{\$83} & \$63 & \$37 & \$75 \end{bmatrix}$$

$\$3 \times 13 + \$4 \times 8 + \$2 \times 6$

In that example we multiplied a 1×3 matrix by a 3×4 matrix (note the 3s are the same), and the result was a 1×4 matrix.

In General:

To multiply an **m×n** matrix by an **n×p** matrix, the **n**s must be the same, and the result is an **m×p** matrix.

$$m \times n \ \times \ n \times p \rightarrow m \times p$$

Order of Multiplication

In arithmetic we are used to:

$$3 \times 5 = 5 \times 3$$

(The Commutative Law of Multiplication)

But this is **not** generally true for matrices (matrix multiplication is **not commutative**):

$$AB \neq BA$$

When we change the order of multiplication, the answer is (usually) **different**.

Example:

See how changing the order affects this multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$$

Identity Matrix

The "Identity Matrix" is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- Its symbol is the capital letter **I**.

It is a **special matrix**, because when we multiply by it, the original is unchanged:

$$A \times I = A$$

$$I \times A = A$$

Determinant of a Matrix

The determinant of a matrix is a **special number** that can be calculated from a square matrix.

A Matrix is an array of numbers:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 2 Columns)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant of that matrix is (calculations are explained later):

3×6 − 8×4 = 18 − 32 = **−14**

What is it for?

The determinant tells us things about the matrix that are useful in systems of linear equations, helps us find the inverse of a matrix, is useful in calculus and more.

Symbol

The **symbol** for determinant is two vertical lines either side.

Example:

|A| means the determinant of the matrix **A**

(Exactly the same symbol as absolute value.)

Calculating the Determinant

First of all the matrix must be **square** (i.e. have the same number of rows as columns). Then it is just basic arithmetic. Here is how:

For a 2×2 Matrix

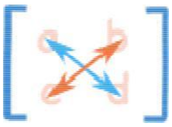
For a 2×2 matrix (2 rows and 2 columns):

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:



- Blue means positive (+ad),
- Red means negative (-bc)

Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$|B| = 4\times8 - 6\times3$$

$$= \quad 32-18$$
$$= \quad 14$$

Cramer's Rule

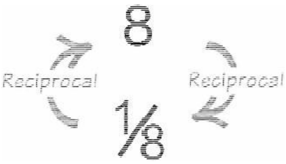
Cramer's Rule, is used to solve systems with matrices. Cramer's Rule was named after the Swiss mathematician Gabriel Cramer, who also did a lot of other neat stuff with math.

Cramer's rule is all about getting determinants of the square matrices that are used to solve systems.

Inverse of a Matrix

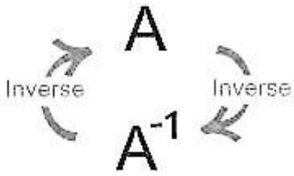
What is the Inverse of a Matrix?

This is the reciprocal of a **Number**:



Reciprocal of a Number

The **Inverse of a Matrix** is the **same idea** but we write it A^{-1}



Why not $^{1}/_{A}$? Because we don't divide by a Matrix! And anyway $^{1}/_{8}$ can also be written 8^{-1}

And there are other similarities:

When you **multiply a number** by its **reciprocal** you get **1**

$$8 \times (^{1}/_{8}) = \mathbf{1}$$

When you **multiply a Matrix** by its **Inverse** you get the **Identity Matrix** (which is like "1" for Matrices):

$$A \times A^{-1} = \mathbf{I}$$

Same thing when the inverse comes first:

$$(^{1}/_{8}) \times 8 = \mathbf{1}$$

$$A^{-1} \times A = \mathbf{I}$$

Identity Matrix

We just mentioned the "Identity Matrix". It is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- It's symbol is the capital letter **I**.

The Identity Matrix can be 2×2 in size, or 3×3, 4×4, etc ...

Definition

Here is the definition:

The Inverse of A is A^{-1} only when:

$$A \times A^{-1} = A^{-1} \times A = \mathbf{I}$$

Sometimes there is no Inverse at all.

2x2 Matrix

OK, how do we calculate the Inverse?

Well, for a 2x2 Matrix the Inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \underset{\substack{\nwarrow \\ \text{determinant}}}{\frac{1}{ad-bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).

Let us try an example:

$$\begin{aligned} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} &= \frac{1}{4\times6-7\times2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \end{aligned}$$

How do we know this is the right answer?

Remember it must be true that: $A \times A^{-1} = \mathbf{I}$

So, let us check to see what happens when we multiply the matrix by its inverse:

$$\begin{bmatrix}4 & 7\\ 2 & 6\end{bmatrix}\begin{bmatrix}0.6 & -0.7\\ -0.2 & 0.4\end{bmatrix} = \begin{bmatrix}4\times0.6 + 7\times-0.2 & 4\times-0.7 + 7\times0.4\\ 2\times0.6 + 6\times-0.2 & 2\times-0.7 + 6\times0.4\end{bmatrix}$$

$$= \begin{bmatrix}2.4 - 1.4 & -2.8 + 2.8\\ 1.2 - 1.2 & -1.4 + 2.4\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

And, hey!, we end up with the Identity Matrix! So it must be right.

It should **also** be true that: $A^{-1} \times A = \mathbf{I}$

Why don't you have a go at multiplying these? See if you also get the Identity Matrix:

$$\begin{bmatrix}0.6 & -0.7\\ -0.2 & 0.4\end{bmatrix}\begin{bmatrix}4 & 7\\ 2 & 6\end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad\qquad \end{bmatrix}$$

Why Would We Want an Inverse?

Because with Matrices we **don't divide**! Seriously, there is no concept of dividing by a Matrix.

But we can **multiply by an Inverse**, which achieves the same thing.

Imagine you couldn't divide by numbers, and someone asked "How do I share 10 apples with 2 people?"

But you could take the **reciprocal** of 2 (which is 0.5), so you could answer:

10 × 0.5 = 5

They get 5 apples each

The same thing can be done with Matrices:

Say that you know Matrix A and B, and want to find Matrix X:

$$XA = B$$

It would be nice to divide both sides by A (to get X=B/A), but remember **we can't divide**.

But what if we multiply both sides by A^{-1} ?

$$XAA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we could remove "1" from 1x = ab for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate A^{-1})

In that example we were very careful to get the multiplications correct, because with Matrices the order of multiplication matters. AB is almost never equal to BA.

A Real Life Example

A group took a trip on a bus, at \$3 per child and \$3.20 per adult for a total of \$118.40.

They took the train back at \$3.50 per child and \$3.60 per adult for a total of \$135.20.

How many children, and how many adults?

First, let us set up the matrices (be careful to get the rows and columns correct!):

$$\overset{\begin{matrix}\text{Child} & \text{Adult}\end{matrix}}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \overset{\begin{matrix}\text{Bus} & \text{Train}\end{matrix}}{\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}} = \overset{\begin{matrix}\text{Bus} & \text{Train}\end{matrix}}{\begin{bmatrix} 118.4 & 135.2 \end{bmatrix}}$$

This is just like the example above:

$$XA = B$$

So to solve it we need the inverse of "A":

$$\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3\times3.6-3.5\times3.2}\begin{bmatrix} 3.6 & -3.5 \\ -3.2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

Now we have the inverse we can solve using:

$$X = BA^{-1}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 118.4 & 135.2 \end{bmatrix}\begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 118.4\times-9 + 135.2\times8 & 118.4\times8.75 + 135.2\times-7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 22 \end{bmatrix}$$

There were 16 children and 22 adults!

The answer almost appears like magic. But it is based on good mathematics.

Calculations like that (but using much larger matrices) help Engineers design buildings, are used in video games and computer animations to make things look 3-dimensional, and many other places.

It is also a way to solve Systems of Linear Equations.

The calculations are done by computer, but the people must understand the formulas.

Order is Important

Say that you are trying to find "X" in this case:

$$AX = B$$

This is different to the example above! X is now **after** A.

With Matrices the order of multiplication usually changes the answer. Do not assume that AB = BA, it is almost never true.

So how do we solve this one? Using the same method, but put A^{-1} in front:

$$A^{-1}AX = A^{-1}B$$

And we know that $A^{-1}A = I$, so:

$$IX = A^{-1}B$$

We can remove I:

$$X = A^{-1}B$$

And we have our answer (assuming we can calculate A^{-1})

Why don't we try our example from above, but with the data set up this way around. (Yes, you can do this, just be careful how you set it up.)

This is what it looks like as AX = B:

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3\times3.6-3.2\times3.5} \begin{bmatrix} 3.6 & -3.2 \\ -3.5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix}$$

It looks so neat! I think I prefer it like this.

Also note how the rows and columns are swapped over ("Transposed")
compared to the previous example.

To solve it we need the inverse of "A":

It is like the Inverse we got before, but

Transposed (rows and columns swapped over).

Now we can solve using:

$$X = A^{-1}B$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix} \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix} = \begin{bmatrix} -9 \times 118.4 + 8 \times 135.2 \\ 8.75 \times 118.4 - 7.5 \times 135.2 \end{bmatrix} = \begin{bmatrix} 16 \\ 22 \end{bmatrix}$$

Same answer: 16 children and 22 adults.

So, Matrices are powerful things, but they do need to be set up correctly!

The Inverse May Not Exist

First of all, to have an Inverse the Matrix must be "Square" (same number of rows and columns).

But also the **determinant cannot be zero** (or you would end up dividing by zero). How about this:

$$\begin{aligned} \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^{-1} &= \frac{1}{3\text{x}8-4\text{x}6} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix} \\ &= \frac{1}{24-24} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix} \end{aligned}$$

24-24? That equals 0, and **1/0 is undefined**.

We cannot go any further! This Matrix has no Inverse.

Such a Matrix is called "Singular", which only happens when the determinant is zero.

And it makes sense ... look at the numbers: the second row is just double the first row, and does **not add any new information**.

Imagine in our example above that the prices on the train were exactly, say, 50% higher ... we wouldn't be any closer to figuring out how many adults and children ... we need something different.

And the determinant neatly works this out.

Conclusion

- The Inverse of A is A^{-1} only when $A \times A^{-1} = A^{-1} \times A = \mathbf{I}$
- To find the Inverse of a 2x2 Matrix: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).
- Sometimes there is no Inverse at all

Algebraic Properties of Matrix Operations

In this page, we give some general results about the three operations: addition, multiplication, and multiplication with numbers, called **scalar multiplication**.

From now on, we will not write (mxn) but mxn.

Properties involving Addition. Let *A*, *B*, and *C* be mxn matrices. We have

1. $A+B = B+A$

2. $(A+B)+C = A + (B+C)$

$$A + \mathcal{O} = A$$

Where $\mathcal{O}$ is the mxn zero-matrix (all its entries are equal to 0);

4. $A + B = \mathcal{O}$ if and only if $B = -A$.

Properties involving Multiplication.

1. Let *A*, *B*, and *C* be three matrices. If you can perform the products *AB*, (*AB*)*C*, *BC*, and *A*(*BC*), then we have

$$(AB)C = A\,(BC)$$

Note, for example, that if *A* is 2x3, *B* is 3x3, and *C* is 3x1, then the above products are possible (in this case, (*AB*)*C* is 2x1 matrix).

2. If α and β are numbers, and *A* is a matrix, then we have

$$\alpha(\beta A) = (\alpha\beta)A$$

3. If α is a number, and *A* and *B* are two matrices such that the product $A \cdot B$ is possible, then we have

$$\alpha(AB) = (\alpha A)B = A(\alpha B)$$

4. If *A* is an nxm matrix and $\mathcal{O}$ the mxk zero-matrix, then

$$A\mathcal{O} = \mathcal{O}$$

Note that $A\mathcal{O}$ is the nxk zero-matrix. So if n is different from m, the two zero-matrices are different.

Properties involving Addition and Multiplication.

1. Let *A*, *B*, and *C* be three matrices. If you can perform the appropriate products, then we have
   (*A*+*B*)*C* = *AC* + *BC* and *A*(*B*+*C*) = *AB* + *AC*

2. If α and β are numbers, *A* and *B* are matrices, then we have

$\alpha(A+B) = \alpha A + \alpha B$	And	$(\alpha+\beta)A = \alpha A + \beta A$

Example. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & 1 & 5 \end{pmatrix}.$$

Evaluate (*AB*)*C* and *A*(*BC*). Check that you get the same matrix.

Answer. We have

$$AB = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

So

$$(AB)C = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -5 \\ 0 & -2 & -10 \end{pmatrix}.$$

On the other hand, we have

$$BC = \begin{pmatrix} 0 & 2 & 10 \\ 0 & -1 & -5 \end{pmatrix}$$

so

$$A(BC) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 10 \\ 0 & -1 & -5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -5 \\ 0 & -2 & -10 \end{pmatrix}.$$

Example. Consider the matrices

$$X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \text{ and } Y = \begin{pmatrix} \alpha & \beta & \nu & \gamma \end{pmatrix}.$$

It is easy to check that

$$X = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$Y = \alpha \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} + \nu \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

These two formulas are called **linear combinations**. More on linear combinations will be discussed on a different page.

We have seen that matrix multiplication is different from normal multiplication (between numbers). Are there some similarities? For example, is there a matrix which plays a similar role as the number 1? The answer is yes. Indeed, consider the nxn matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In particular, we have

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix I_n has similar behavior as the number 1. Indeed, for any nxn matrix A, we have

$$A\, I_n = I_n\, A = A$$

The matrix I_n is called the **Identity Matrix** of order n.

Example. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}.$$

Then it is easy to check that

$$AB = I_2 \text{ and } BA = I_2.$$

The identity matrix behaves like the number 1 not only among the matrices of the form nxn. Indeed, for any nxm matrix A, we have

$$I_n A = A \text{ and } AI_m = A.$$

In particular, we have

$$I_4 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Applications of Matrix Mathematics

Matrix mathematics applies to several branches of science, as well as different mathematical disciplines. Let’s start with computer graphics, then touch on science, and return to mathematics.

We see the results of matrix mathematics in every computer-generated image that has a reflection, or distortion effects such as light passing through rippling water.

Before computer graphics, the science of optics used matrix mathematics to account for reflection and for refraction.

Matrix arithmetic helps us calculate the electrical properties of a circuit, with voltage, amperage, resistance, etc.

In mathematics, one application of matrix notation supports graph theory. In an adjacency matrix, the integer values of each element indicates how many connections a particular node has.

The field of probability and statistics may use matrix representations. A probability vector lists the probabilities of different outcomes of one trial. A stochastic matrix is a square matrix whose rows are probability vectors. Computers run Markov simulations based on stochastic matrices in order to model events ranging from gambling through weather forecasting to quantum mechanics.

Matrix mathematics simplifies linear algebra, at least in providing a more compact way to deal with groups of equations in linear algebra.

Daily Matrix Applications

Matrix mathematics has many applications. Mathematicians, scientists and engineers represent groups of equations as matrices; then they have a systematic way of doing the math. Computers have embedded matrix arithmetic in graphic processing algorithms, especially to render reflection and refraction. Some properties of matrix mathematics are important in math theory.

CHAPTER 2: NUMBER SYSTEM AND BINARY ARITHMETIC

Introduction to Number System

Definition: Number system is the way to represent a number in different forms.

Types of Number system:

1. **Binary Number System:** It is the number system with base value 2 means it has only two digits to represent the data. The digits are (0, 1). E.g. 00,01,10,11,100….
2. **Decimal Number System:** It is the number system with base value 10 means it has 10-digits to represent the data. The digits are(0-9). Eg. 0,1,2,3,4,5,6 ………
3. **Octal Number System:** It is the number system with base value 8 means it has 8 digits to represent the data. The digits are (0-7).
4. **Hexadecimal Number System :** It is the number system with base value 16 means it has 16 digits to represent the data. The digits are (0-15). Eg. 0,1,2,3…….,9,A,B,C,D,E,F

Bits & Bytes:

1 bit(binary digit) = 1 **digit**. For example: 1

1 byte = 8-**bits**

1 kilo byte= 2^{10} = 1024 **bytes**

1 mega byte = $2^{10} * 2^{10} = 2^{20}$ = 1024 **kilo bytes**

1 giga byte= 2^{30}= 1024 **mega bytes**

1 tera byte= 2^{40}= 1024 **giga bytes**

Binary Number System:

In binary number system is made up of 2 digits- 0 and 1. We use these two digits to represent data.

0	Start at 0
1	Then 1
???	Then no other symbol

So we count in the same way as using decimal number system. For example:

Decimal number system start at 0 and then 1,2,3,4,5,6,7,8,9… now what after nine repeat the no in combination such as start at 0 again the add 1 to the left of 0 resultant 10 , 11 ,12… so on. 100, 1000 etc.

The same method we follow in Binary number system:

Decimal Number	Binary Value	Decimal Number	Binary Value
0	0	6	110
1	1	7	111
2	10	8	1000
3	11	9	1001
4	100	10	1010
5	101	11	1011

So confused!
how we get this?

Lets do a little Mathematics:

	2^5	2^4	2^3	2^2	2^1	2^0	=
	32	16	8	4	2	1	✓
0=	0	0	0	0	0	**0**	
1=	0	0	0	0	0	**1**	
2=	0	0	0	0	**1**	**0**	
3=	0	0	0	0	**1**	**1**	
4=	0	0	0	**1**	**0**	**0**	
5=	0	0	0	**1**	**0**	**1**	
6=	0	0	0	**1**	**1**	**0**	
7=	0	0	0	**1**	**1**	**1**	
8=	0	0	**1**	**0**	**0**	**0**	
9=	0	0	**1**	**0**	**0**	**1**	
10=	0	0	**1**	**0**	**1**	**0**	

Confuse again? Have a look on the bold letter 2, 1 is representing that 2

STOP? Are you able to create the binary counting

If no, Just look at the power values : 2^0, 2^1, 2^2, 2^3, 2^4…….2^n

2^0= 1

2^1= 2

2^3= 8

2^4=16 so on…..

Lets calculate 2, we have only two digits **0** and **1.**

For 2 , I can write in front of 2^1 as its equal to 2.

Hmmm…..again what about 3? If I will add 2+1 =3 so there for 1 is assigned in front of the power of 2^0 and 2^1.
Still not get : Ok read this,

Decimal

Well how do we count in Decimal?	0	Start at 0
	...	Count 1,2,3,4,5,6,7,8, and then...
	9	This is the **last digit** in Decimal
	10	So we start back at 0 again, but add 1 on the left

The same thing is done in binary ...

	Binary	
	0	Start at 0
•	1	Then 1
••	10	Now start back at 0 again, but **add** 1 **on the left**
•••	11	1 more
••••	???	But NOW what ... ?

Other Number System

The octal system

In this system the base is eight. The allowed digits are 0 – 7 where as 8 is not allowed.

Typical number

N= $(4526.23)_8$

In polynomial term may be represented as follow

$(4\times8^3)+(5\times8^2)+(2\times8^1)+(6\times8^0)+(2\times8^{-1})+(3\times8^{-2})$ - POLYNOMIAL TERM

= $(2390.296875)_{10}$ = $(2390\ 19/64)_{10}$ - DECIMAL TERM

The hexadecimal

The base here is 16 and the allowed digits are 0 – 9 and A – F

A typical number

$N = (A1F.1C)_{16}$

In Polynomial may be represented as

$(A\times16^{2})+ (1\times16^{1}) + (F\times16^{0}) + (1\times16^{-1}) + (C\times16^{-2}) =$

$(10\times16^{2}) + (1\times16^{1})+(15\times16^{0}) + (1\times16^{-1}) + (12\times16^{-2}) =$

$(2591\ 28/256)_{10} = 2591.109375_{10}$

Conversion between Number System

To convert a decimal number into binary number system. Follow the following Steps:-

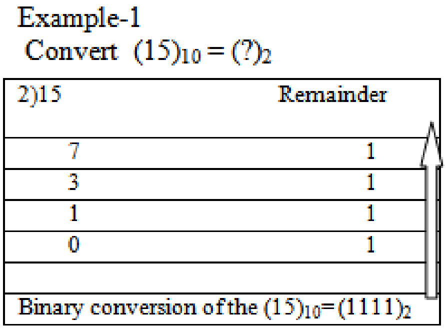
Step-1: Divide the Number by 2 (as 2 is the base of the binary number system).

Step-2 : Collect the remainder.

Step-3: Divide the quotient again with 2.

Step-4: Repeat the step 2 & 3 until the quotient is 0.

Step-5: Start from bottom, read the sequence of remainders upwards to the top.

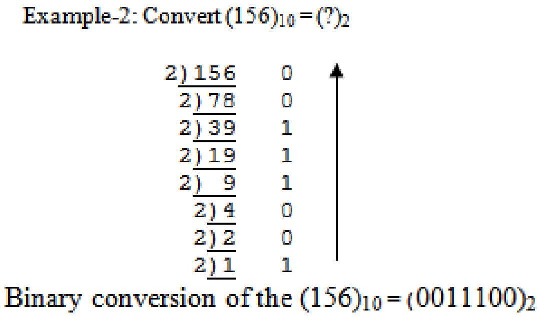


Example-1

Convert $(15)_{10} = (?)_2$

2)15	Remainder
7	1
3	1
1	1
0	1

Binary conversion of the $(15)_{10}=(1111)_2$



Example-2: Convert $(156)_{10}=(?)_2$

2) 156	0
2) 78	0
2) 39	1
2) 19	1
2) 9	1
2) 4	0
2) 2	0
2) 1	1

Binary conversion of the $(156)_{10} = (0011100)_2$

Hexadecimal Number system:

This number system has a base value 16. We can use (0-15) decimal numbers to represent hexadecimal numbers. It uses 16 distinct numbers to represent the values.

Decimal numbers	Binary Number	Hexadecimal Number
0	00	0
1	01	1
2	10	2
3	11	3
4	100	4
5	101	5
6	110	6
7	111	7
8	1000	8
9	1001	9
10	1010	A
11	1011	B
12	1100	C
13	1101	D
14	1110	E
15	1111	F

ASCII (American Standard Code Interchange Information) Code: **ASCII** is a character set that is used to interchange information to binary and from binary to decimal. ASCII is a 8-bit character containing 256 characters.

0-31	Control code
32-127	Standard, alphabet A-Z and a-z
128-255	Special Symbols, non standard characters

Unicode: Unicode is a character set that is used to interchange information to binary language and from binary to decimal language. The latest version of Unicode is **Unicode 6.0**. It is a computing industry standard encoding scheme to represent a text.

To convert number from a non-decimal to decimal, we simply expand a given number as a polynomial and evaluate the polynomial using arithmetic as in the examples above.

When a decimal number is converted to any other system, the integer and the fraction potions of the number are handled differently. The radix divide technique is used to convert the integer portion and the radix multiply technique is used for the fraction portion.

Example

1. $(245)_{10}$ to binary

2	245		
2	122	1	LSB
2	61	0	
2	30	1	
2	15	0	
2	7	1	
2	3	1	
2	1	1	
	0	1	MSB

$N = 11110101_2$

2. 0.625_{10} to binary

0.625×2=	1.250	0.100	MSB
0.250×2=	0.500	0.000	
0.500×2=	1.000	0.001	LSB

$N = 0.101_2$

3. 245_{10} to hexadecimal

16	245		
16	5	15	MSB
2	0	5	LSB

$N = 5F_{16}$

4. $2AF_{16}$ to decimal
   $(2\times16^2)+(A\times16^1)+(F\times16^0) = (2\times16^2)+(10\times16^1)+(15\times16^0)$

   $= (512+160+15)_{10} = 687_{10}$

To convert hexadecimal to binary umber, simply replace each hexadecimal bit with its 4 bit equivalent binary bit

i.e. – 37_{16} = 00110111

- $C4_{16}$ = 11000100

To convert a binary number to its hexadecimal equivalent, simply group the binary bits at groups of 4. if necessary, may have to add 0's to complete the groups

Note

The leading zero that is added to complete the MSB assist us in making 4 bit binary group.

The grouping of a 4 bit binary number is referred as ***binary coded hexadecimal***

Introduction to Binary Arithmetic

Arithmetic circuits form point of the CPU. Mathematical operations include

Subtraction, multiplication, division and addition

Addition

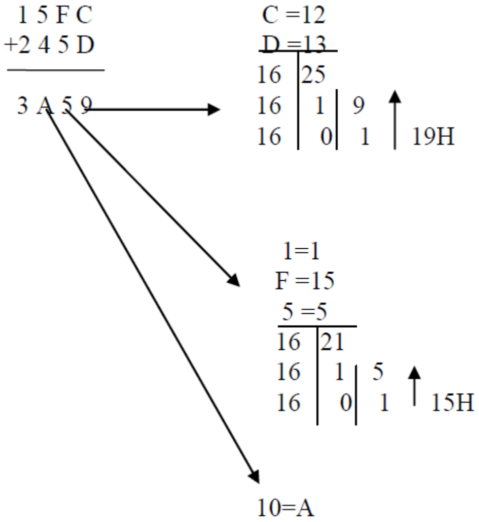
a) Binary addition
Binary addition takes in consideration of the following conditions

0+0=0, 0+1=1, 1+0=1,1+1=0

When adding larger numbers, the resulting ones are carried to other higher columns e.g.

```
 101        1010
+010  and  +0011
-----      -----
 111        1101
```

b) hexadecimal addition
Let’s add



Alternatively ADD the binary equivalence of the hexadecimal numbers

1	5	F	C
0001	0101	1111	1100
2	4	5	D
0010	0100	0101	1101
3	A	5	9
0011	1010	0101	1001

Subtraction

a) Binary subtraction

In arithmetic subtraction, the initial numeric quotients that are combined by subtraction are the minuend and the subtrahend.. the result of the subtraction is called the difference

A → Minu-end
-B → Subtra-end
―――
C → Difference

To subtract from a larger binary number, subtract column by column borrowing from adjacent columns when necessary.

Remember when borrowing from adjacent column, there are two digits

<u>Example</u>

1. subtract 1001 from 1101

1001
1101
―――
0100

2. subtract 0111 from 1011

0111
1011
―――
0100

When subtracting, 0 becomes 2

Binary numbers can also be -ve

The procedure for this calculation is identical to that of decimal numbers because the smaller value is subtracted from the larger value and the negative sign placed in front of the results

111	4
100	7
100	-3

There are two other methods available for doing subtraction and representation of a negative number.

i. ones' compliment

The procedure for subtracting numbers using ones' compliment is as follows

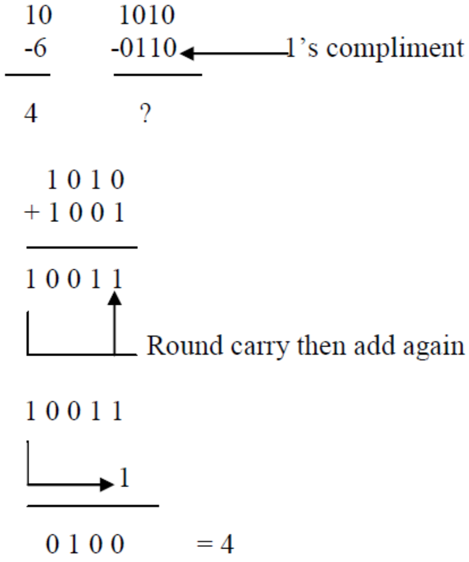
Step1-change the 0's of subtra-end to 1's and the 1's of subtra-end to 0's

step2-add the two numbers

Step3-remove the last carry and add it to the number

i.e. – end round carry

<u>Example</u>



When there is a carry in the end of the result, then we know the result is positive.

When there is no carry then we know the result is negative and we can now place a minus sign in front of the answer.

Example

01101 → menuend

→

11011 → subtraend

00100 1’s compliment of subtraend

01101 →

→

+ 00100

ii. two’s compliment

The general rule is that the two’s complement subtra-end is added to menu-end.

If the sign bit of the result is 0, then the result is the true difference and is assigned a positive sign.

If the sign bit is 1, the result is a two’s complement of the difference.

Any over flow produced by the calculation is lost.

Example

Subtract 11011 from 01101

Store 11011

Reverse bits 00100 1’s compliment

Add one + 1

00101 2’s compliment =

01101

+00101

10010

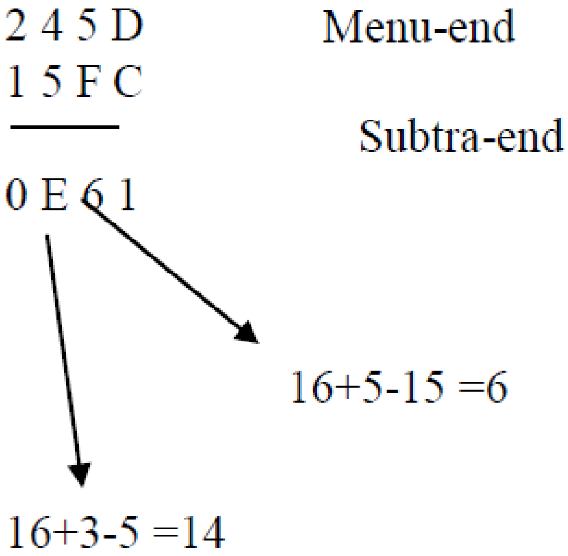
\- 1

10001 → reverse → 01110 (14)

can now add a minus -01110 (-14)

b) Hexadecimal subtraction
Example

Subtract 15FCH from 245DH



Alternatively convert the decimal numbers to binary then subtract using the rules of binary subtraction.

Multiplication

c) Binary multiplication
Binary numbers are multiplied in the same manner as decimal numbers.

Rules

- 0×0=0, 0×1=0, 1×0=0, 1×1=1

- To multiply number with more than one digit, you form partial products and add them together.

$$\begin{array}{r} 5 \\ \times 6 \\ \hline 30 \end{array} \qquad\qquad \begin{array}{r} 101 \\ \times 110 \\ \hline 000 \\ 101 \\ 101 \\ \hline 11110 \end{array}$$

Computers cannot store partial products. The multiplication method used by the computer is **repeated additions**.

To determine 7×55, the computer can add 7, 55 times

A faster method of micro-processor system is the **add and shift method**

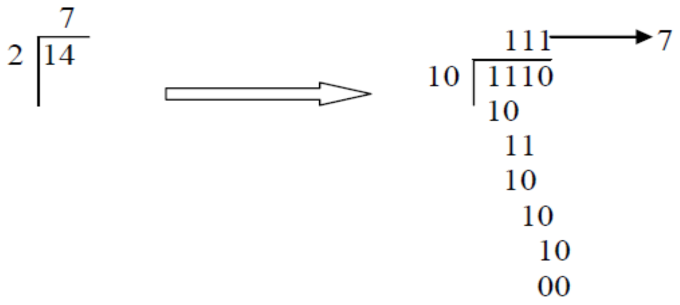
Division

d) Binary division

There are several methods of performing binary division. In the partial method also known as the restoring method, division is similar to decimal method

Example

Divide 14 by 2



Successive subtraction

Divisor is subtracted from the divided and from each successive remainder until a barrow is realized. The desired quotient is one less the number of subtraction needed to produce a borrow. This method is simple but slow for large numbers.

Representation of negative numbers

The examples shown so far have been using positive numbers. In practice, a digital system must represent all positive and negative numbers. To accommodate the sign of numbers, an additional digit known as the sign digit is included in the representation along with the magnitude digit. Thus to represent an n-digit number, we world need n+1 digit.

Typically, the sign of digit is the MSB

There are two ways to represent sign numbers

e) sign-magnitude system

in this representation, n+1 digit are used to represent a number where the MS digit is the sign digit and the remaining n-digit are the magnitude digit. The value of the sign digit is 0 for a positive number and r-1 for a negative number, where r is the radix of the number system.

Sign	Magnitude			Decimal
0	1	1	1	+7
0	1	1	0	+6
0	1	0	1	+5
0	1	0	0	+4
0	0	1	1	+3
0	0	1	0	+2
0	0	0	1	+1
0	0	0	0	0
1	0	0	1	-1
1	0	1	0	-2
1	0	1	1	-3
1	1	0	0	-4
1	1	0	1	-5
1	1	1	0	-6
1	1	1	1	-7

The sign and magnitude portions are handled separately in arithmetic using sign magnitude number

f) Compliment system

To compliment a binary number, change all 0's to 1's and all 1's to 0's. this is known as ones' compliment form of a binary number.

i.e. - 0110 = 1001 in compliment

The most common way to express a negative binary number is to show it as a two's compliment. A two's compliment is a binary number that shows when one is added to the first compliment

Signed Decimal	Ones' compliment	Two's compliment
+7	0111	0111
+6	0110	0110
+5	0101	0101
+4	0100	0100
+3	0011	0011
+2	0010	0010
+1	0001	0001
0	0000	0000
1	1110	1111
2	1101	1110
3	1100	1101
4	1011	1100
5	1010	1101
6	1001	1010
7	1000	1001

Using the two's compliment makes its easier for digital system to perform digital operation. The correct sign bit is generated by performing the two's compliment

Computer data formats/coding

Successful programming requires a precise data format.

Data appears in different forms as below

a) Alphanumeric codes

Codes have been developed to represent data as well as numbers and special symbols as '=', '&', etc

The codes are called alphanumeric codes

The American Standard Code for Information Interchange (ASCII) is a seven bit code that is commonly used in computer systems.

Since ASCII is a seven bit code, there are 2^7 or 128 possible coding combinations which mean that each letter of alphabet both upper and lower case as well as decimal digits 0 – 9 and special characters are represented by a unique code.

i.e. – A – 100 001, 0 – 011 0000 and & - 010 1010

Many window based applications use the Unicode system to store alphanumeric data. This systems store each character as 16 bit data

The code 0000H – 00FFH is the same as the standard ASCII code

b) Binary Coded Decimal (BCD)

The BCD system provides a convenient way of handling large numbers that need to be input/output from a microcomputer system.

The BCD system provides a means of converting a code readily handled by human (decimal) to a code readily handled by machine (binary).

LED displays are examples that may use BCD system.

BCD system uses four bits to represent each decimal digit. The four bits used are the binary equivalent of the numbers 0 – 9

The binary representation of a decimal number is obtained by replacing each decimal digit by its BCD equivalent

c) Float point numbers

Fixed point representation is convenient for representing numbers with bounded order s of magnitude

i.e. – in a digital computer that uses 32 bits to represent numbers, the range of integers that can be used is limited to $\pm (2^{31}-1) \approx \pm 10^{11}$

In scientific computing environment, a wide range of numbers may be needed. In such a case float-point representation is used.

The general form of a float-point representation is given by

$$N = F \times R^E$$

Where

N – Number

F – Fraction or mantissa

R – Radix

E – Exponent

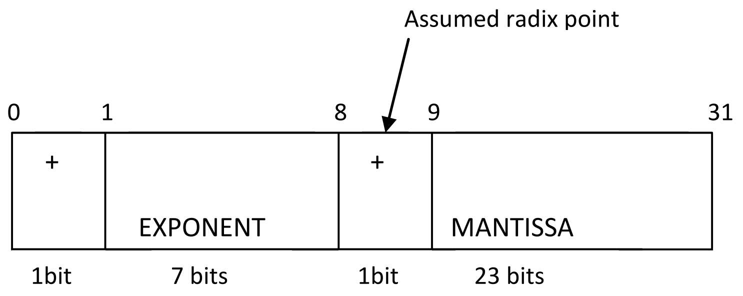
Consider the number N=3560000, there are several ways of writing same number

$N = 0.356\times10^7 = 0.0356\times10^8 = 3.56\times10^6$ etc

All these forms are valid float point representation.

We can remove the zeros in the first two forms and the integer will still be valid in which case the resulting form requires the fewest integers to represent the mantissa since all significant zeros have been eliminated from the mantissa. This is known as the normalized form of float point representation.

Float point numbers are generally stored in the memory in the format shown below



Note

The MSB of the exponent is the exponent sign bit and the MSB of the mantissa is the mantissa sign bit

e) Other Binary Coding

Binary codes are codes which are represented in binary system with modification from the original ones. Below we will be seeing the following:

- Weighted Binary Systems
- Non Weighted Codes

Weighted Binary Systems

Weighted binary codes are those which obey the positional weighting principles, each position of the number represents a specific weight. The binary counting sequence is an example.

Decimal	8421	2421	5211	Excess-3
0	0000	0000	0000	0011
1	0001	0001	0001	0100
2	0010	0010	0011	0101
3	0011	0011	0101	0110
4	0100	0100	0111	0111
5	0101	1011	1000	1000
6	0110	1100	1010	1001
7	0111	1101	1100	1010
8	1000	1110	1110	1011
9	1001	1111	1111	1100

8421 Code/BCD Code: The BCD (Binary Coded Decimal) is a straight assignment of the binary equivalent. It is possible to assign weights to the binary bits according to their positions. The weights in the BCD code are 8,4,2,1.

Example: The bit assignment 1001, can be seen by its weights to represent the decimal 9 because:

1x8+0x4+0x2+1x1 = 9

2421 Code: This is a weighted code, its weights are 2, 4, 2 and 1. A decimal number is represented in 4-bit form and the total four bits weight is 2 + 4 + 2 + 1 = 9. Hence the 2421 code represents the decimal numbers from 0 to 9.

5211 Code : This is a weighted code, its weights are 5, 2, 1 and 1. A decimal number is represented in 4-bit form and the total four bits weight is 5 + 2 + 1 + 1 = 9. Hence the 5211 code represents the decimal numbers from 0 to 9.

Reflective Code: A code is said to be reflective when code for 9 is complement for the code for 0, and so is for 8 and 1 codes, 7 and 2, 6 and 3, 5 and 4. Codes 2421, 5211, and excess-3 are reflective, whereas the 8421 code is not.

Sequential Codes: A code is said to be sequential when two subsequent codes, seen as numbers in binary representation, differ by one. This greatly aids mathematical manipulation of data. The 8421 and Excess-3 codes are sequential, whereas the 2421 and 5211 codes are not.

Non Weighted Codes

Non weighted codes are codes that are not positionally weighted. That is, each position within the binary number is not assigned a fixed value.

Excess-3 Code: Excess-3 is a non-weighted code used to express decimal numbers. The code derives its name from the fact that each binary code is the corresponding 8421 code plus 0011(3).

Example: 1000 of 8421 = 1011 in Excess-3

Note

Excess-3 binary-coded decimal **(XS-3)**, also called **biased** representation or **Excess-*N***, is a numeral system used on some older computers that uses a pre-specified number *N* as a biasing value. It is a way to represent values with a balanced number of positive and negative numbers. In XS-3, numbers are represented as decimal digits, and each digit is represented by four bits as the BCD value plus 3 (the "excess" amount):

- The smallest binary number represents the smallest value. (i.e. 0 − Excess Value)
- The greatest binary number represents the largest value. (i.e. 2^{N+1} − Excess Value − 1)

Decimal	Binary	Decimal	Binary	Decimal	Binary	Decimal	Binary
−3	0000	**1**	0100	**5**	1000	**9**	1100
−2	0001	**2**	0101	**6**	1001	**10**	1101
−1	0010	**3**	0110	**7**	1010	**11**	1110
0	0011	**4**	0111	**8**	1011	**12**	1111

To encode a number such as 127, then, one simply encodes each of the decimal digits as above, giving (0100, 0101, 1010).

The primary advantage of XS-3 coding over BCD coding is that a decimal number can be nines' complemented (for subtraction) as easily as a binary number can be ones' complemented; just invert all bits.

Adding Excess-3 works on a different algorithm than BCD coding or regular binary numbers. When you add two XS-3 numbers together, the result is not an XS-3 number. For instance, when you add 1 and 0 in XS-3 the answer seems to be 4 / 3 instead of 1. In order to correct this problem, when you are finished adding each digit, you have to subtract 3 (binary 11) if the digit is less than decimal 10 and add three if the number is greater than or equal to decimal 10 (thus causing the number to wrap).

Gray Code: The gray code belongs to a class of codes called minimum change codes, in which only one bit in the code changes when moving from one code to the next. The Gray code is non-weighted code, as the position of bit does not contain any weight. The gray code is a reflective digital code which has the special property that any two subsequent numbers codes differ by only one bit. This is also called a unit-distance code. In digital Gray code has got a special place.

Decimal Number	Binary Code	Gray Code
0	0000	0000
1	0001	0001
2	0010	0011
3	0011	0010
4	0100	0110
5	0101	0111
6	0110	0101
7	0111	0100
8	1000	1100
9	1001	1101
10	1010	1111
11	1011	1110
12	1100	1010
13	1101	1011
14	1110	1001
15	1111	1000

Binary to Gray Conversion

- Gray Code MSB is binary code MSB.
- Gray Code MSB-1 is the XOR of binary code MSB and MSB-1.
- MSB-2 bit of gray code is XOR of MSB-1 and MSB-2 bit of binary code.
- MSB-N bit of gray code is XOR of MSB-N-1 and MSB-N bit of binary code.

CHAPTER 3: LOGIC GATES AND BOOLEAN ALGEBRA

Introduction to Logic Mathematics

Mathematical logic is a subfield of **mathematics** exploring the applications of formal **logic** to **mathematics**. It bears close connections to meta-mathematics, the foundations of **mathematics**, and theoretical computer science.

Set theory

A ***set*** can be defined as a collection of ***things*** that are brought together because they obey a certain ***rule***.
These 'things' may be anything you like: numbers, people, shapes, cities, bits of text ..., literally anything.

The key fact about the 'rule' they all obey is that it must be *well-defined*. In other words, it enables us to say for sure whether or not a given 'thing' belongs to the collection. If the 'things' we're talking about are English words, for example, a well-defined rule might be:

> '... has 5 or more letters'

A rule which is not well-defined (and therefore couldn't be used to define a set) might be:

> '... is hard to spell'

Requirement of a set

1. A set must be well defined i.e. it must not leave any room for ambiguities e.g sets of all students- which? Where? When?

A set must be defined in terms of space and time

2. The objective (elements or members) from a given set must be distinct i.e each object must appear once and only once, Must appear but not more than once

3. The order of the presentation of elements of a given set is immaterial
   e.g 1,2,3 = 1,3,2 = 3,2,1

Types of Sets

In set theory, there are different types of sets. All the operations in set theory could be based on sets. Set should be a group of individual terms in domain. The universal set has each and every element of domain. We are having different types of sets. We will see about the different types of sets.

Different Types of Sets

There are different types of sets in set theory. They are listed below:

- Universal Set
- Empty set
- Singleton set
- Finite and Infinite set
- Union of sets
- Intersection of sets
- Difference of sets
- Subset of a set
- Disjoint sets
- Equality of two sets

Universal Set

The set of all the 'things' currently under discussion is called the ***universal set*** (or sometimes, simply the ***universe***). It is denoted by **U**.

The universal set doesn't contain everything in the whole universe. On the contrary, it restricts us to just those things that are relevant at a particular time. For example, if in a given situation we're talking about numeric values – quantities, sizes, times, weights, or whatever – the universal set will be a suitable set of numbers (see below). In another context, the universal set may be {alphabetic characters} or {all living people}, etc.

Singleton Set:

A set which contains only one element is called a singleton set.

For example:

- A = {x : x is neither prime nor composite}

  It is a singleton set containing one element, i.e., 1.

- B = {x : x is a whole number, $x < 1$}

  This set contains only one element 0 and is a singleton set.

- Let A = {x : $x \in N$ and $x^2 = 4$}

  Here A is a singleton set because there is only one element 2 whose square is 4.

- Let B = {x : x is a even prime number}

  Here B is a singleton set because there is only one prime number which is even, i.e., 2.

Finite Set:

A set which contains a definite number of elements is called a finite set. Empty set is also called a finite set.

For example:

- The set of all colors in the rainbow.
- N = {x : $x \in N$, $x < 7$}
- P = {2, 3, 5, 7, 11, 13, 17, ... 97}

Infinite Set:

The set whose elements cannot be listed, i.e., set containing never-ending elements is called an infinite set.

For example:

- Set of all points in a plane
- A = {x : x ∈ N, x > 1}
- Set of all prime numbers
- B = {x : x ∈ W, x = 2n}

Note:

All infinite sets cannot be expressed in roster form.

For example:

The set of real numbers since the elements of this set do not follow any particular pattern.

Cardinal Number of a Set:

The number of distinct elements in a given set A is called the cardinal number of A. It is denoted by n(A).

For example:

- A {x : x ∈ N, x < 5} A = {1, 2, 3, 4}
  Therefore, n(A) = 4

- B = set of letters in the word ALGEBRA

  B = {A, L, G, E, B, R}

  Therefore, n(B) = 6

Equivalent Sets:

Two sets A and B are said to be equivalent if their cardinal number is same, i.e., n(A) = n(B). The symbol for denoting an equivalent set is '↔'.

For example:

A = {1, 2, 3} Here n(A) = 3 B = {p, q, r} Here n(B) = 3 Therefore, A ↔ B

Equal sets:

Two sets A and B are said to be equal if they contain the same elements. Every element of A is an element of B and every element of B is an element of A.

For example:

A = {p, q, r, s} B = {p, s, r, q}

Therefore, A = B

The various types of sets and their definitions are explained above with the help of examples.

Empty Set

In mathematics, empty set is a set theory related topic. A set without any elements is said to be an empty set. This article helps you understand empty set by giving a clear idea about empty set with some example problems.

Empty Set Definition

The other name of empty set is null set ϕ. Consider two sets X = {a, b, c, d} and Y = {1, 2, 3, 4, 5, 6}. Consider another set Z which represents the intersection of X and Y. There is no common element for the set X and Y. So, intersection of X and Y is null.

Z = { } **The representation of empty set is { }.**

Empty Set or Null Set:

- A set which does not contain any element is called an empty set, or the null set or the void set and it is denoted by ∅ and is read as phi. In roster form, ∅ is denoted by {}. An empty set is a finite set, since the number of elements in an empty set is finite, i.e., 0.

- **For example:** (a) The set of whole numbers less than 0.
  (b) Clearly there is no whole number less than 0.

  Therefore, it is an empty set.

  (c) N = {x : x ∈ N, 3 < x < 4}

  • Let A = {x : 2 < x < 3, x is a natural number}

  Here A is an empty set because there is no natural number between 2 and 3.

  • Let B = {x : x is a composite number less than 4}.

  Here B is an empty set because there is no composite number less than 4.

  Note:

  ∅ ≠ {0} ∴ has no element.

  {0} is a set which has one element 0.

  The cardinal number of an empty set, i.e., n(∅) = 0

Cardinality of Empty Set:

Since we know that the cardinal number represents the number of elements that are present in the set and by the definition of an empty set, we know that there are no element in the empty set. Hence, the cardinal number or cardinality of an empty is zero.

Properties of Preparation for Empty Set:

1. Empty set is considered as subset of all sets. $\phi \subset X$
2. Union of empty set ϕ with a set X is X. $A \cup \phi = A$

Intersection of an empty set with a set X is an empty set.

Solved Examples

Question 1: A is a set of alphabets and B is a set of numbers. What is the intersection of A and B?

Solution: A ∩ B = { }

Question 2: Write the set A which is a set of goats with 10 legs.

Solution: A = { }

Power Set of the Empty Set

A set is called the power set of any set, if it contains all subsets of that set. We can use the notation P(S) for representing any power set of the set. Now, from the definition of an empty set, it is clear that there is no element in it and hence, the power set of an empty set i.e.P(ϕ) is the set which contain only one empty set, hence P(ϕ) = {ϕ}

Cartesian Product Empty Set

The Cartesian product of any two sets say A and B are denoted by A X B. There are some conditions for Cartesian product of empty sets as follows:

If we have two sets A and B in such a way that both the sets are empty sets, then A X B = ϕ x ϕ = ϕ. It is clear that, the cartesian product of two empty sets is again an empty set.

If A is an empty set and B = {1, 2, 3}, then the cartesian product of A and B is as follows: A X B = {ϕ}. {1, 2, 3} = {ϕ X 1, ϕ X 2, ϕ x 3} = {ϕ, ϕ, ϕ} = {ϕ}

So, we say that if one of the set is an empty set from the given two sets, then again the Cartesian product of these two sets is an empty set.

Examples of Empty Sets

Given below are some of the examples of empty sets.

Solved Examples

Question 1: Which of the following represents the empty set?

1. A set of cats with 4 legs
2. A set of apples with red color
3. A set of positive numbers in which all are less than 1
4. A set of rectangles with 4 sides

Solution:

Option 1: A set of cats with 4 legs. This set is possible where cats are having 4 legs.

Option 2: A set of apples with red color. This set is possible where apple is in red color.

Option 3: A set of positive numbers in which all are less than 1.

This set is not possible because the positive numbers must be greater than 1. So, this set is considered as empty set.

Answer: 3

Question 2: A is a set of numbers from 1 to 10 B is a set of negative numbers. What is the intersection of A and B?

Solution:

Given:

A = set of number from 1 to 10. = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} B = set of negative numbers

= {-1, -2, -3, -4,….} Intersection of A and B = A ∩ B = { }

Answer: The intersection of given sets is an empty set.

Subset

Consider the sets, X = set of all students in your school and Y = set of all students in your class. It is obvious that set of all students in your class will be in your school. So, every element of Y is also an element of X. We say that Y is a subset of X. The fact that Y is a subset of X is expressed in symbol as $Y \subset X$. The symbol ⊂ stands for "is a subset of" or "is contained in".If Y is a subset of X, then X is known to be a superset of Y. The subset of a set will have elements equal to or less than the elements in the given set.

Subset Definition

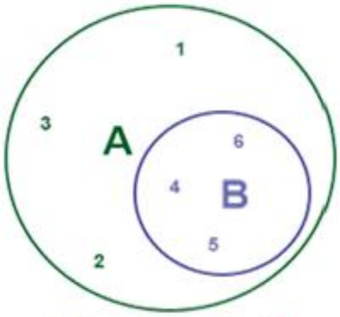
A set A is said to be a subset of a set B, if every element of A is also an element of B. In other words, $A \subset B$ if whenever $a \in A$, then $a \in B$. It is often convenient to use the symbol ⇒ which means "implies". Suppose, for two sets A and B, A = {1, 2, 3} and B = {1} then B is the subset of A.

Subset Symbol:

Using the symbol ⇒, we can write the definition of subset as follows:

$A\subset B$ if $a\in A\Rightarrow a\in B$

We read it as "A is a subset of B if a is an element of A, which implies that a is also an element of B". If A is not a subset of B, we write A is not a subset of B. If A = {1, 2, 3, 4, 5, 6} and B = {4, 5, 6}, then we can draw a Venn diagram for this as follows:



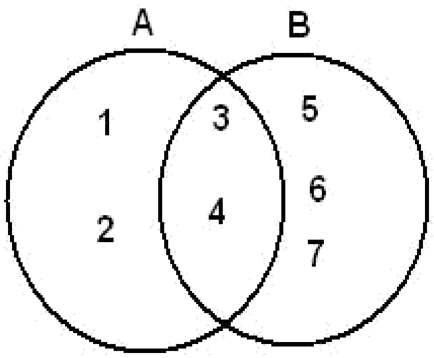
B is subset of A

Operation of a Set

Union of Sets

Set is an important part of the mathematics. It is applied in almost many branch of mathematics. Set is the relation of some givenn data. There are many functions of set like union, intersection. Here, we will discuss about union of sets.

We denote the union of A and B by $A\cup B$. Thus, $AUB=x|x\in A\text{or}x\in B\text{or}x\in A\text{and}B$. We write $AUB=x|x\in A\text{or}x\in B$ where, it is unnderstood that the word 'or' is used in the inclusive sense. That is, $x\in A$ or $x\in B$ stands for $x\in A$ or $x\in B$ or $x\in A\text{and}B$.

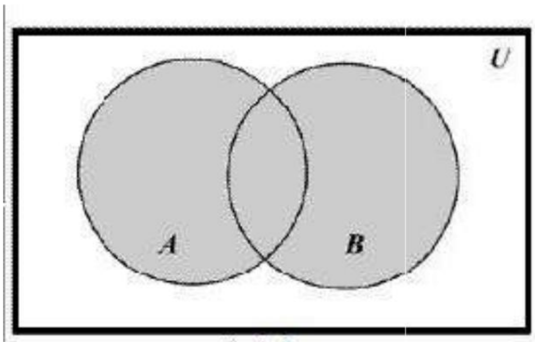


Union of Two Sets

Let we have two sets A and B, thhen the union of these two sets is the set of all elements of each sets i.e. the set of those elements which are in either sets.

If A = {1,2,3,4} and B = {3,4,5,66,7} then $A \cup B$ = {1, 2, 3, 4, 5, 6, 7}.

With the help of Venn diagram, we can prove it.



A ⋃ B

Union of Countable Sets

A set of natural numbers which is a subset of a set with the same number of elements is called the countable set. The union of two countable sets is again a countable set. Let X and Y be two countable sets then X ∪ Y is countable. Clearly, if X ∪ Y is countable, then X and Y are each countable, as they are subsets of a countable set.

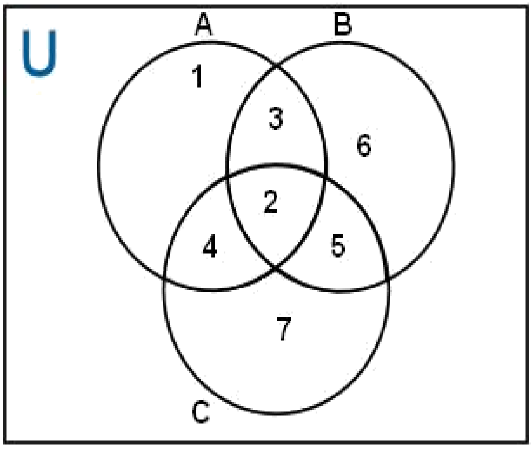
Conversely, let us suppose that we have two countable sets X and Y. And, we can define two surjection functions f: N → X and g : N → Y. Let Z = X ∪ Y. Then, we can define h : N → Z in a way that h(2n + 1) = f(n) for n = 0, 1, ... and h(2n) = g(n), n = 1, 2,... Then, h is well defined function for every value of i ∈ N is either odd or even, so h(i) is defined. Since h is onto function for any z ∈ Z, then z ∈ X or z ∈ Y. If z ∈ X, then h(2q + 1) = z for some value of q and if z ∈ Y then h(2p) = z for some value of p. Hence, Z is countable. So, we can say that the union of two countable sets is again a countable set.

Union of Three Sets

If we have three sets say A, B and C, then the union of these three sets is the set that contains all the elements or all contains that belongs to either A or B or C or to all three sets.

A = {1, 2, 3, 4}, B = { 2, 3, 5, 6} and C = { 2, 4, 5, 7}. Then, $A \cup B \cup C$ = { 1, 2, 3, 4, 5, 6, 7}

We can show it in the Venn diagram as follows:



Union of Sets Examples

Given below are some of the examples on union of sets.

Solved Examples

Question 1: Find the union of each of the following two sets:

1. X = {1, 3, 6} Y = {1, 2, 6}
2. X = {a, e, i, o, u} Y = {a, e, c}
3. X = {3 , 4 ,5} B = φ

Solution:

$X \cup Y$ = {1, 2, 3, 6}

$X \cup Y$ = {a, c, e, i, p, u}

$X \cup Y$ = {3, 4, 5}

Question 2:

If X = {1, 2, 5, 6}, Y = {3, 4, 6, 9}, Z = {3, 5, 6, 9} and W = {3, 6, 9, 11}. Find

1. $X \cup Y$
2. $X \cup Z$
3. $Y \cup Z$
4. $Y \cup W$
5. $X \cup Y \cup Z$
6. $X \cup Y \cup W$
7. $Y \cup Z \cup W$

Solution:

1. $X ∪ Y$ = {1, 2, 3, 4, 5, 6,9}
2. $X ∪ Z$ = {1, 2, 3, 5, 6, 9}
3. $Y ∪ Z$ = {3, 4, 5, 6, 9}
4. $Y ∪ W$ = {3, 4, 5, 6, 9,11}
5. $X ∪ Y ∪ Z$ = {1, 2, 3, 4, 5, 6, 9}
6. $X ∪ Y ∪ W$ = {1, 2, 3, 4, 5, 6, 9, 11}
7. $Y ∪ Z ∪ W$ = {3, 4, 5, 6, 9,11}

Find the Union of the Sets

Here, we will learn how to find the union of the sets with the help of the following examples.

Solved Examples

Question 1:

Two sets are given.

A = {5, 12, 13, 16, 19}

B = {5, 10, 13, 16, 19}

Find $A ∪ B$

Solution:

Given sets are:

A = {5, 12, 13, 16, 19}

B = {5, 10, 13, 16, 19}

$A ∪ B$ = {5, 10, 12, 13, 16, 19}

Here, common elements in A, B are 5,13,16,19

So, it is taken only one times.

Question 2:

Find $X ∪ Y$ for the following set.

X = {4, 6, 8, 9, 11}

Y = {3, 5, 6, 8, 11}

Solution:

Given sets are

X = {4, 6, 8, 9, 11}

Y = {3, 5, 6, 8, 11}

So, $X \cup Y$ = {3, 4, 5, 6, 8, 9, 11}

Here, common element is taken only one time.

Intersection of Sets

Intersection is an operation on sets. It is just opposite to union. It is a very useful and important concept in set theory. Before we learn about intersection, we need to understand some basic concept like what is set.

A set is a well-defined collection of data. It's data is known as it's members or elements. We represent the set by capital letters A, B, C, X, Y, Z, etc. We use the concept of set in daily life. For example, a team has five members. So, this is a set.

Find the Intersection of the Sets

For finding the intersection of two sets, we usually select those elements which are common in both the sets. If there are three sets, then we select those elements which are common in all three sets. Hence, if there are n number of sets, then we select only those elements which are common in all the n sets. In this way, we find the intersection of sets

Intersecting Set: Two sets A and B are said to be intersecting if A $\cap$ B $\neq$ ϕ

Disjoint set: Two sets A and B are said to be disjoint if A $\cap$ B = ϕ

Solved Examples

Question 1:

If A = {1, 3, 4, 6, 9} and B = {2, 4, 6, 8}, find A ∩ B. What do you conclude?

Solution:

We have given that A = {1, 3, 4, 6, 9} and B = {2, 4, 6, 8}

We have to find the intersection of A and B.

So, A ∩ B = {1, 3, 4, 6, 9} ∩ {2,4,6,8}

A ∩ B = {4, 6}

Question 2:

If A = {1, 3, 5, 7, 9} and B = {2,4,6,8}, find A ∩ B. What do you conclude?

Solution:

We have A ∩ B = {1, 3, 5, 7, 9} ∩

{2,4,6,8} = ϕ

If no data match in both the sets, both the sets are known as **disjoint sets**. Thus, A and B are disjoint sets.

Question 3:

If A = {1, 2, 3, 4, 5, 6, 7}, B = {2, 4, 6, 8, 10} and C = {4, 6, 7, 8, 9, 10, 11}, then find A ∩ B and A ∩ B ∩ C.

Solution:

Given sets are

A = {1, 2, 3, 4, 5, 6, 7}

B = {2, 4, 6, 8, 10}

C = {4, 6, 7, 8, 9, 10, 11}

First, we have to find A ∩ B. Then, we have to treat A ∩ B as a single

set. For A ∩ B, we select those elements which are common in sets A and

B. So, A ∩ B = {2, 4, 6}

For (A ∩ B) ∩ C, we select those elements which are common in sets A ∩ B and C.

So, (A ∩ B) ∩ C = {4, 6}

So, A ∩ B ∩ C = {4, 6}

Question 4:

If A = {1,3,5,7,9}, B = {2,4,6,8} and C = {2,3,5,7,11}, find (A ∩ B) and (A ∩ C) What do you conclude?

Solution:

We have given that

A = {1, 3, 5, 7, 9}

B = {2, 4, 6, 8}

C = {2, 3, 5, 7, 11}

A ∩ B = {1, 3, 5, 7, 9} ∩ {2, 4, 6, 8} = ϕ

Thus, A and B are disjoint sets

A ∩ C = {1,3,5,7,9} ∩ {2,3,5,7,11} = {3,5,7}

Thus, A and B are disjoint sets while A and C are intersecting sets.

Intersection of Two Sets

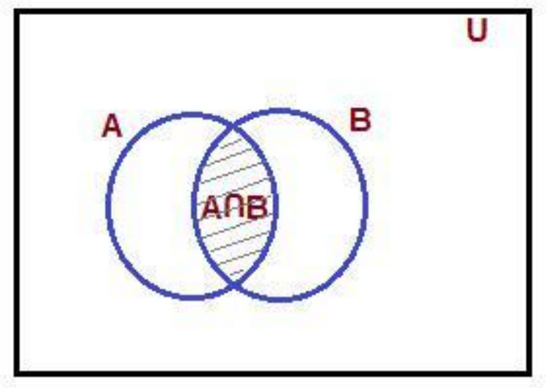
The intersection of two sets is the set of all the elements of two sets that are common in both of them. If we have two sets A and B, then the intersection of them is denoted by A $\cap$ B and it is read as A intersection B.

Let X = {2, 3, 8, 9} and Y = {5, 12, 9, 16} are two sets.

Now, we are going to understand the concept of **Intersection of set.** It is represented by the symbol " $\cap$ ".

If we want to find the intersection of A and B, the common part of the sets A and B is the intersection of A and B. It is represented as A $\cap$ B. That is, if an element is present in both A and B, then that will be there in the intersection of A and B. It will be more clear with the below figure.

Let A and B are two sets. Then, the intersection of A and B can be shown as below.



The intersection of A and B is denoted by A $\cap$ B.

Thus, A $\cap$ B = {x : x $\in$ A and x $\in$ B}.

Clearly, x $\in$ A $\cap$ B i.e., x $\in$ A and x $\in$ B

In the above figure, the shaded area represents A $\cap$ B.

In the same way, if A_1, A_2, ..., A_n is a finite family of sets, then their intersection is represented by A_1 $\cap$ A_2 $\cap$......$\cap$ A_n.

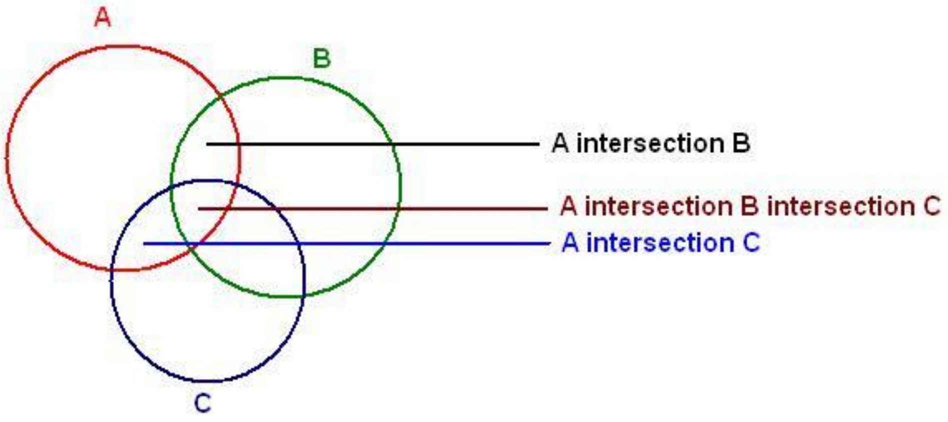
Intersection of Convex Sets

In a Vector space, a set is called convex set if all the elements of the line joining two points of that set also lies on that set. In other words, we can say that the set S is convex set if for any points x, y $\in$ S, there are no points on the straight line joining points x and y are not in the set S.

The intersection of two convex set is again a convex set. We can prove it with the help of contradiction method. So, lets suppose that A and B are the two convex sets. And, let we have two points x and y in such a way that x ∈ A $\cap$ B and y $\in$ A $\cap$ B, then x $\in$ A, x $\in$ B, y $\in$ A and y $\in$ B and there exists a point z in such a way that z is not in A or B or both. This is the contradiction of our assumption that A and B are the convex sets. So there is no such point x, y and z can exists and A $\cap$ B is a convex set.

Intersection of Three Sets

If we have A, B and C, then the intersection of these three sets are the set of all elements A, B and C that are common in these three sets.



Solved Example

Question:

If we have A = {1, 3, 5, 7, 6, 8}, B = {2, 4, 6, 8, 9} and C = {1, 3, 6, 8 }, then find the A ∩ B ∩ C.

Solution:

Given that A = {1, 3, 5, 7, 6, 8}, B = {2, 4, 6, 8, 9} and C = {1, 3, 6, 8 }.

Then, it is clear that the elements 6 and 8 are common in all the three given sets.

Hence, we get A ∩ B ∩ C = {6, 8}.

Intersection of Open Sets

Every intersection of open sets is again an open set. Let us have two open sets A_1 and A_2. If the intersection of both of them is empty and empty set is again an open set. Hence, the intersection is an open set.

If A_1 and A_2 are open sets, then there exists some x $\in$ A_1 $\cap$ A_2. Since the given sets are open, we have some r_1 and r_2 in such a way that $B_{r_{1}}(x)\subset A_{1}$ and $B_{r_{2}}(x)\subset A_{2}$. So, we can choose a number $B_{r}(x)\subset A_{1}\cap A_{2}$.

So, we can say that if the intersection is not empty, then by the use of definition of intersection and non emptiness, there exists any x $\in$ A_i for all A_i's, where all A_i's are open sets. Then, we have $B_{r_{i}}(x)\subset A_{i}$ for some $r_i > 0$.

Complement of a Set

In set theory, complement set is one of the branch. Set of all elements in the universal set that are not in the initial set are said to be complement set. The complement of a set is represented by the symbol A'. The set is a collection of the object. Set is denoted by the symbols {}. In this article, we see in detail about the complement set.

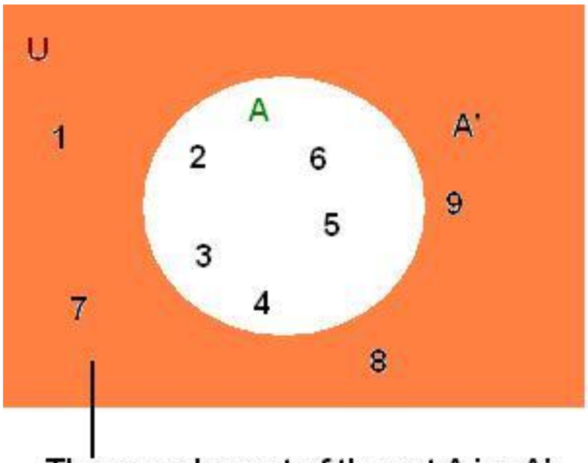
Complement of a Set Definition

If we have a set A, then the set which is denoted by U - A, where U is the universal set is called the complement of A. Thus, it is the set of everything that does not belong to A. So, the complement of a set is the set of those elements which does not belong to the given set but belongs to the universal set U. Mathematically, we can show it as $A^c = \{x \setminus x \notin A \text{ but } x \in U\}$

Since we know that every set is the subset of the universal set U, then the complementary set is also the subset of U. The total number of elements in the complementary set is equal to the difference between the number of elements of the set U and the number of elements of the given set (say A). If A is the given set, then the complement of A is denoted as A^c or A'.

For example, U = { 1, 2, 3, 4, 5, 6, 7, 8, 9} and a set A = { 2,3,4,5,6}. Then, the complement of A is denoted by A^c or A'.

A^c = {1, 7, 8, 9}. We can show this with the help of Venn diagram



The complement of the set A i.e. A'

Complement of a Set Example

Given below are some of the examples on complement of a set.

Solved Examples

Question 1: Value of set U = {2, 4, 6, 7, 8, 9, 10} and A = {7, 8, 9, 10} and B = {8, 9, 10}. Find the complement of A, complement of B, complement of A union B.

Solution:

Step 1: Given

U = {2, 4, 6, 7, 8, 9, 10}

A = {7, 8, 9, 10}

B = {8, 9, 10}

Step 2: The element of set U is {2, 4, 6, 7, 8, 9, 10}. The element that does not belong to A is {2, 4, 6}. Complement of A is {2, 4, 6}.

Step 3: Complement of B is {2, 4, 6, 7}

Step 4: Complement of AB is {2, 4, 6, 8, 9, 10}.

Question 2: Values of set U = {3, 5, 7, 8, 9, 10, 12} and A = {8, 9, 10, 12}. Find the compliment of A.

Solution:

Step 1: Given

U = {3, 5, 7, 8, 9, 10, 12}

A = {8, 9, 10, 12}

Step 2: The element of set U is {3, 5, 7, 8, 9, 10, 12}. Elements {3, 5, 7} does not belong to the set A. So, A' = {3, 5, 7}

Step 3: Complement of A is {3, 5, 7}.

Question 3: Values of set U = {1, 4, 6, 7, 8, 10} and A = {6, 7, 8}. Find the complement of A

Solution:

Step 1: Given

U = {1, 4, 6, 7, 8, 10}

A = {6, 7, 8}

Step 2: The element of set U is {1, 4, 6, 7, 8, 10}. Elements {1, 4, 10} does not belong to the set A. A' is {1, 4, 10}.

Step 3: Complement of A is {1, 4, 10}.

Set Difference

Here, we are going to learn about an operation on set called difference of sets. In mathematics, a set can have a limited number of elements. Set is a collection of data. We can perform many operations on set. The difference operation is one of them. The subtract(difference) symbol in the function represents the removal of the values from the second set from the first set. The operation of subtraction is a removing or taking away objects from group of object.

Difference of Two Sets

Difference of sets is defined as a method of rearranging sets by removing the elements which belong to another set. Difference of sets is denoted by either by the symbols - or \. P minus Q can be written either **P - Q** or **P \ Q**.

The differences of two sets P and Q, is written as P - Q, **It contains elements of P which are not present in elements of *Q***. Here, result *P* - *Q* is obtained. Take set P as usual and compare with set Q. Now, remove those element in set P which matches with set Q. If P = {a,b,c,d} and Q = {d,e}, then P - Q = {a,b,c}.

Definition for difference of sets

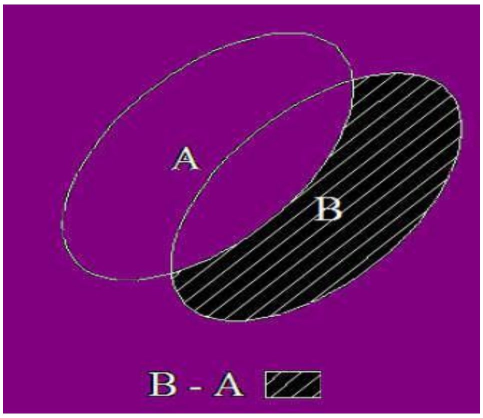
The difference between two sets A and B are represented in the order as the set of all those elements of A which are not in B. It is denoted by A - B.

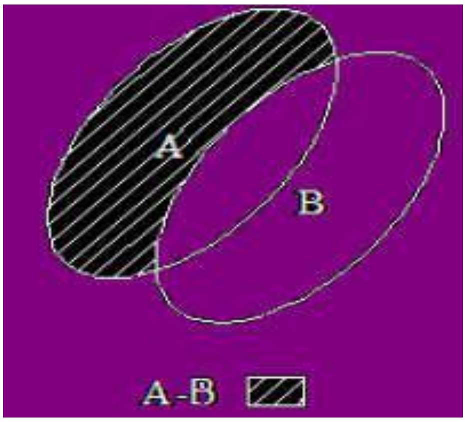
In symbol, we write it as

A - B = {x: x $\in$ A and x $\notin$ B}

Similarly B - A = {x: x $\in$ B and $\notin$ A}.

By representing it in the Venn diagram,

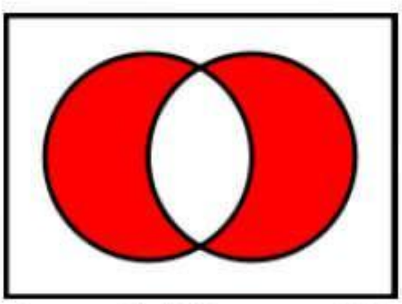




Symmetric Difference of Sets

If we have two sets A and B, then the symmetric difference of these two sets A and B is the set of all elements those are either in A or in B not in both sets. So, we can say that the symmetric difference of two sets is the union without the intersection. We can use the symbol $\triangle$ for this and denoted as follows:

$A \triangle B = \left \{x \setminus x \in A \wedge x \notin B \right \} \vee \left \{ x \notin A \wedge x\in B \right \}$



Symmetric Difference of Sets

$A\triangle B$

The symmetric difference of sets is associative. So, if we have three sets A, B and C, then $(A \triangle B) \triangle C = A \triangle(B \triangle C)$

The symmetric difference of two sets is commutative i.e. for all sets A and B, we have $A \triangle B = B \triangle A$

Set Difference Examples

Given below are some of the problems based on difference of sets.

Solved Examples

Question 1: Consider the two sets A = {11,12, 13, 14, 15, 16}, B = {12, 14, 16, 18}. Find the difference between the two sets?

Solution:

Given A = {11,12, 13, 14, 15, 16}

B = {12, 14, 16, 18}

A - B = {11, 13, 15}

B - A = {18}

The set of all elements are present in A or in B. But, not in both is called the symmetric difference set.

Question 2: A = {2,3,4,1,8,9} and B = {2,3,4,1,8,,12}. What is A - B and B - A?

Solution:

Given A = {2,3,4,1,8,9}

B = {2,3,4,1,8,12}

Here, all elements of A is available in B except 9.

So, the difference A - B = {9}.

Here, all elements of B are available in A except 12.

So, the difference B - A = {12 }.

Question 3: Consider two sets A = {a, b, f, g, h}, B = {f, g, a, k}. Find A - B and B - A?

Solution:

Given A = {a, b , f , g , h}

B = {f, g, a, k} So, A - B = {b, h} and B - A = {k}

Question 4: Consider given sets P = {19, 38, 57, 76, 95} and Q = {7, 19, 57, 75, 94}. Find P - Q and Q - P.

Solution:

Given P = {19, 38, 57, 76, 95}

Q = {7, 19, 57, 75, 94} So, P - Q = {38,76,95} and Q - P = {7,75,94}

Venn Diagrams

In mathematics, we can use the graphs and diagrams to solve some problems in geometry as well as in algebra. To follow this procedure, we can show some relations in set theory with the help of diagram, which is called as the **Venn diagram**. It is also known as **set diagram**. Venn diagrams are named so in the name of its founder John Venn in around 1880.

In set theory, Venn diagrams are studied. A set is defined as a collection of the same types of things. Venn diagram is an important and unique way of representing sets and various operations on them. It is a pictorial representation of sets. It is an easy way to understand about set theory. Venn diagrams are everywhere in set theory. With the help of Venn diagrams, we are able to show the operations of union, intersection, difference, complement etc. on the given sets.

In this page, we can discuss about these things with the help of a Venn diagram. In this process, the sets are represented by circles. Venn diagrams are generally used to represent operations on two or three sets. In order to learn about set theory in detail, one needs to command on Venn diagrams. In this article, students will learn about different types of Venn diagrams. So, go ahead with us and understand about Venn diagrams in detail.

What is a Venn Diagram?

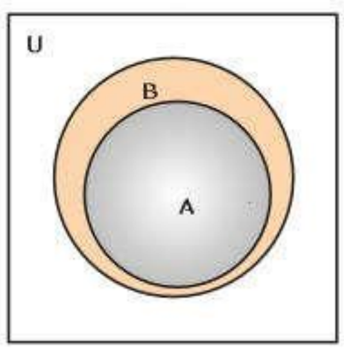
A Venn diagram is a pictorial representation of sets by set of points in the plane. The universal set U is represented pictorially by interior of a rectangle and the other sets are represented by closed figures viz circles or ellipses or small rectangles or some curved figures lying within the rectangle.

Venn diagram is a graphical tool in which we use overlapping circles to visually presentation among some given sets information. In Venn diagram, we can use two or more than two circles to show sets.

Make a Venn Diagram

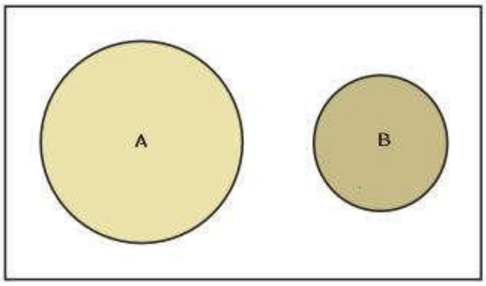
To make a Venn diagram, first we draw a rectangle to show the universal set U and mark U inside the rectangle. After that, we will make circles for given sets and name them as A, B, C etc. Then, according to the given relation of the sets, we can make a diagram for these sets in the rectangle to show the relationship of the sets. Sometimes, we have some elements for the individual sets, then fill all the elements in their respective sets and as per the given relation of the sets.

For example, if A and B are any two arbitrary sets, elements such that, some elements are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B, we represent A and B in the pictorial form as in shown in the Venn diagram.



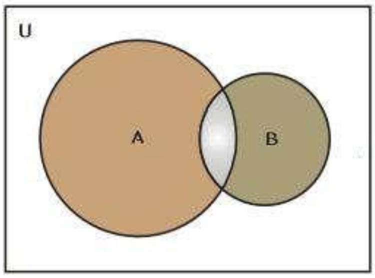
$A \subseteq B$

A is a subset of B and is represented as shown in the Venn diagram.



Disjoint Sets

A and B are disjoint sets as shown in the Venn diagram.

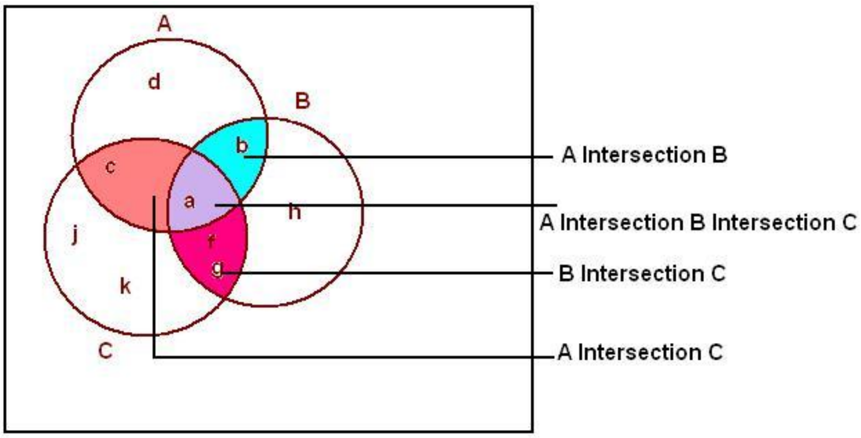


Triple Venn Diagram

For the triple Venn diagram, we need three sets as A, B and C. In the triple Venn diagram, we have to show some relationship between these three sets.

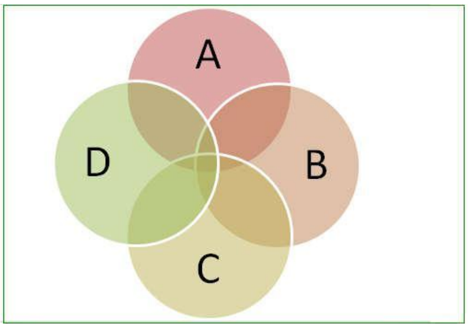
For example, let A = {a, b, c, d, e}, B = {a, b, f, g, h} and C = {a, c, e, f, g, j, k}. Here, we can find A ∩ B, B ∩ C, A ∩ C and A ∩ B ∩ C with the help of triple Venn diagram.

Given A = {a, b, c, d, e}, B = {a, b, f, g, h} and C = {a, c, f, g, j, k}. Now, A ∩ B = {b}, B ∩ C = {f, g}, A ∩ C = {c} and A ∩ B ∩ C = {a}



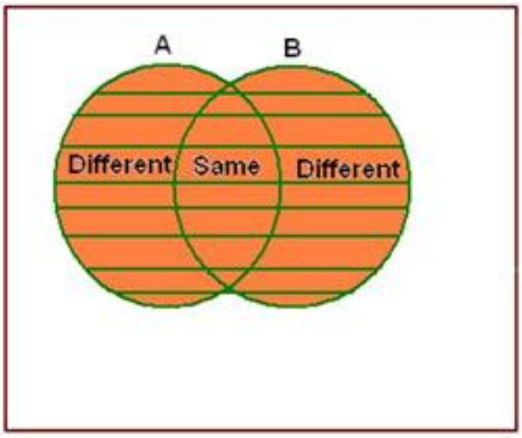
4 Circle Venn Diagram

Some times, we have four sets in a given problem and we want to show their relationship with the help of Venn diagram. For this, we can draw four circles in a rectangle box, each circle represents a unique set. Then, according to sets relation fill all the elements at their place.



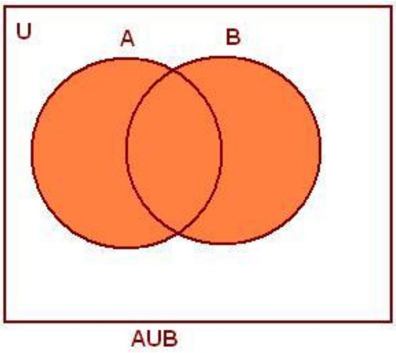
Venn Diagram With Lines

In mathematics, sometimes we use the lines in the Venn diagram to show the union, intersection, difference etc. for the given sets. If we have sets A and B, then with the line Venn diagram we can show as:

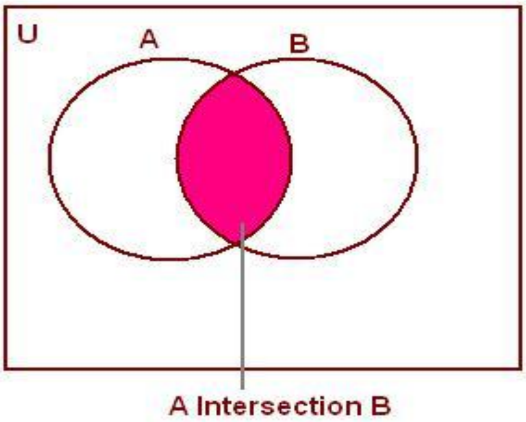


Picture of a Venn Diagram

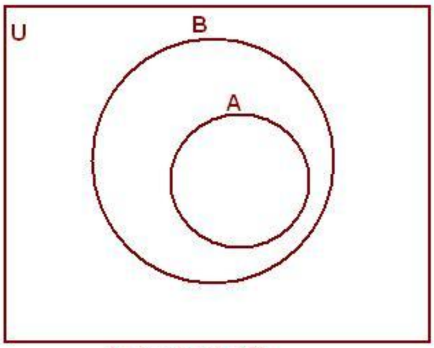
If we have two sets A and B, then **A ∪ B i.e. A union B:**



A ∩ B i.e. A intersection B:



A and B are disjoint sets:



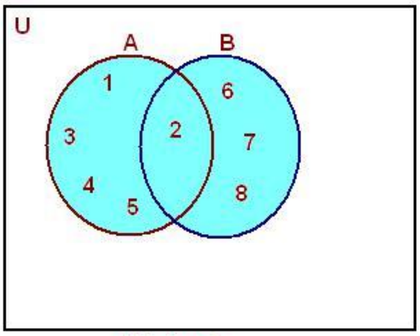
A is subset of B

A subset B:

Venn Diagram Union

If we have two sets A and B, then A ∪ B is the set of all elements that are in set A and in the set B. If any element common in these two set, then we will take that one only one time. So, we can say that the union of the set A and B is everything which are either in set A or in the set B.

Let A = {1,2,3,4,5} and B = {2,6,7,8} then A ∪ B = {1,2,3,4,5,6,7,8}. To show this union, we can use the Venn diagram also as



A Union B

Venn diagram Word Problems

Given below are some of the word problems on Venn diagram.

Solved Example

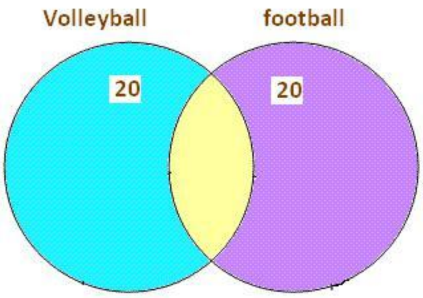
Question: There are 40 players participated in tournament match. In that, 20 players play in volley ball match and 20 players play in football match and 5 players play in both volley ball and football match. Solve this problem by using Venn diagram. How many of the players are either in match and how many are in neither match?

Solution:

There are two categories, one is volleyball and other one is football.

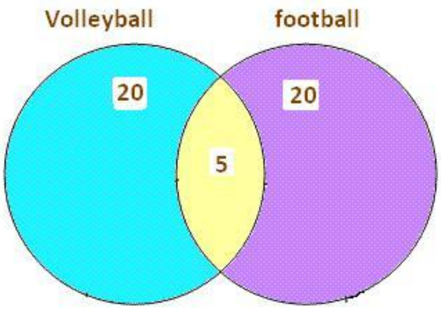
Step 1:

Draw Venn diagram depending up on the classification given in the problem.



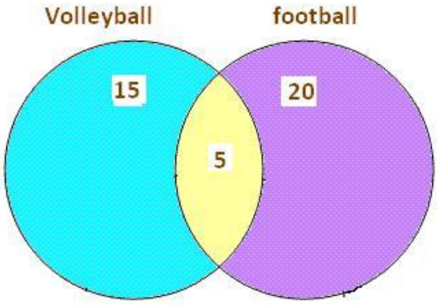
Step 2:

Note that 5 players play both volleyball and football match



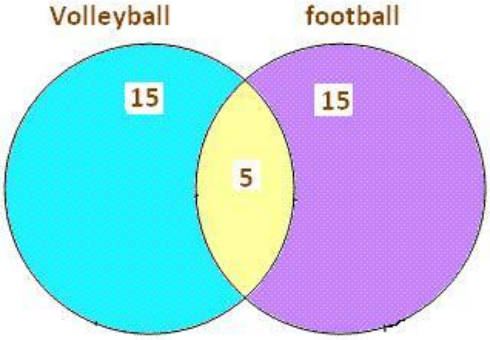
Step 3:

Here, we accounted for 5 of the 20 players in volleyball match, leaving 15 players taking volleyball match but not football match. So, I will put "15" in the "volleyball only" part of the "volley ball" circle.



Step 4:

Here, we accounted for 5 of the 20 players in football match, leaving 15 players taking football match but not volleyball match. So, I will put "15" in the "football only" part of the "football" circle.



Step 5:

The total of 5 + 15 + 15 = 35 players are in either volley ball match or football match (or both). The total numbers of players are 40 and participating players are 35 only.

40 - 35 = 5 players

Boolean Algebra

In 1850, George Boole, an English mathematician developed rules and theorems that became Boolean algebra.

Boole's work was an outcrop of work in physiology called LOGIC.

Logic can be used to break down complex problems to simple and understandable problems.

The binary nature of logic problems was studied by Cloude Shannon of MTI in 1938. Shannon applied Boolean algebra to relay logic switching circuits as means of realizing electric circuits.

Electric circuits used for digital computers are designed to generate only two voltage levels

Eg – high level (≈ 5V) and low level (≈ 0V)

The binary number system requires two symbols hence its logical to identify a binary symbol with each voltage level. If we interpolate the high level as a binary 1and low level as a binary 0, then we are using a positive logic system.

Terminologies in Boolean Algebra

logic function and logic gates

Logic circuit - A computer switching/electronic circuit that consists of a number of logic gates and performs logical operations on data

A logic gate is an idealized or physical device implementing a Boolean function; that is, it performs a logical operation on one or more binary inputs, and produces a single binary output. A logic gate is a small transistor circuit, basically a type of amplifier, which is implemented in different forms within an integrated circuit. Each type of gate has one or more (most often two) inputs and one output.

Boolean operation is any logical operation in which each of the operands and the result take one of two values, as "true" and "false" or "circuit on" and "circuit off."

A Boolean Function is a description of operation (logic operation) on algebraic expression called **Boolean expression** which consists of binary variables, the constants 0 and 1, carried out in digital/electronic circuits and the logic outputting there off. The logic operation is well expressed in truth tables.

Truth tables

A truth table is a breakdown of a logic function by listing all possible values the function can attain. Such a table typically contains several rows and columns, with the top row representing the logical variables and combinations, in increasing complexity leading up to the final function.

Logic Functions gates and circuitry

From Boolean algebra, we get three basic logic factions that form the basis of all digital computer functions. These basic functions are: AND, OR and NOT

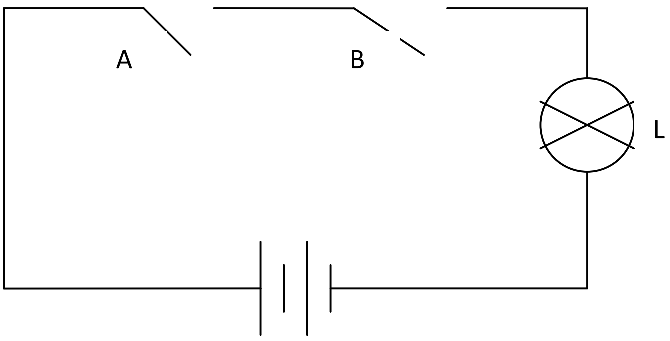
These functions can be expressed mathematically using Boolean algebra as given.

NOTE – The input and output variables are usually represented by letters as ABC or XYZ

- The logic state of this variables is represented by binary numbers 0 and 1

AND function

The AND function can be thought of as a series circuit containing two or more switches



Circuit diagram

The logic indicator L will be ON only when logic switches A and B are both crossed. Switches A and B have two possible logic states, open and crossed. This can be represented in binary form as 0 – open and 1 – crossed.

Logic indicator L also have two possible states 0 and 1

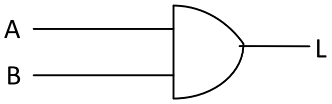
Truth table

A	B	L(x.y)
0	0	0
0	1	0
1	0	0
1	1	1

The truth table is used to illustrate all the possible combinations of input and output conditions that can exist in a logic circuit. The Boolean expression used to represent an AND function is as follows

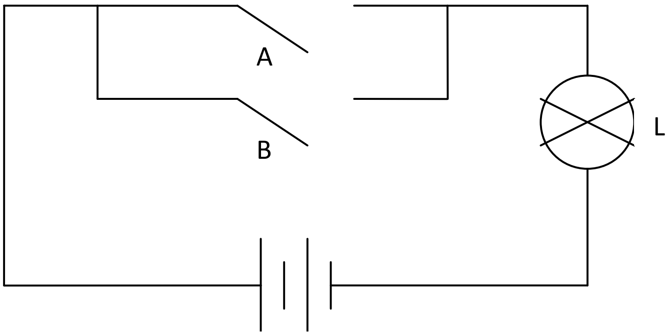
A.B=L

And is symbolized as



OR function

The function can be thought of as a parallel circuit containing two or more logic switches



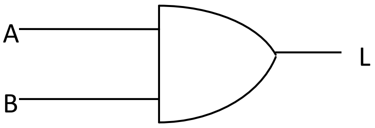
Circuit Diagram

Here, the logic indicator L will be ON whenever logic switch A and B are crossed. The truth table, Expression and Symbol of OR function is as follows

Truth table

A	B	L(x+y)
0	0	0
0	1	1
1	0	1
1	1	1

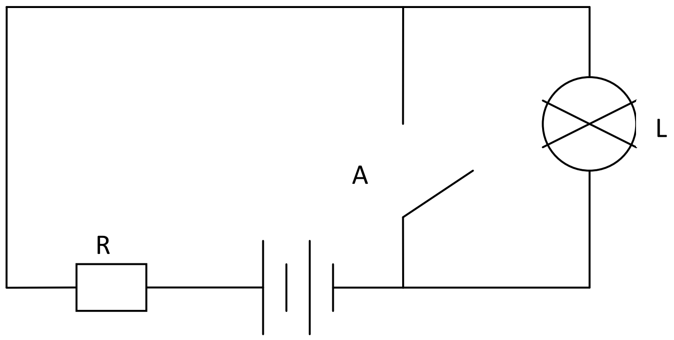
A+B=L



Symbol diagram

NOT function

It can be thought of as an inverter or negative circuit.



Circuit diagram

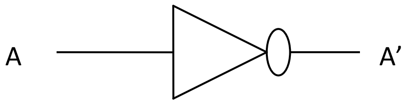
The logic indicator L will be ON whenever logic switch A is open.

The truth table, Expression and Symbol of NOT function is as follows

Truth table

A	L(x)’
0	1
1	0

A =A’



Symbol diagram

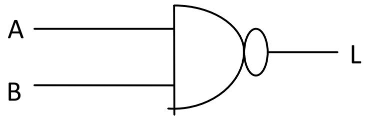
NAND

If an AND gate is followed by an NOT gate then the combination is called an NAND gate and has following truth table and Boolean expression.

Truth table

A	B	L (x.y)’
0	0	1
0	1	0
1	0	0
1	1	0

(A.B)’=L



Symbol diagram

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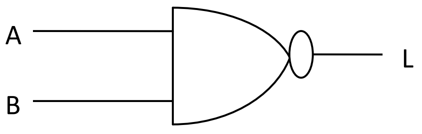
NOR

If an OR gate is followed by an NOT gate then the combination is called an NOR gate and has following truth table and Boolean expression.

Truth table

A	B	L (x+y)’
0	0	1
0	1	1
1	0	1
1	1	0

(A+B)’=L



Symbol diagram

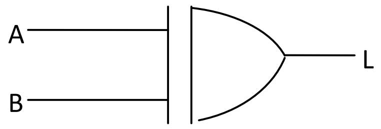
XOR

This output strictly on condition that input is either high but not 2 highs

Truth table

A	B	L (x±y)
0	0	0
0	1	1
1	0	1
1	1	0

$A \oplus B = L$



Symbol diagram

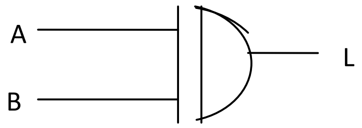
XNOR

This output strictly on condition that input is either high but not 2 highs

Truth table

A	B	L (x±y)’
0	0	1
0	1	0
1	0	0
1	1	1

$A \circledast B = L$



Theorems of Boolean Algebra

Boolean algebra deals with algebraic expressions between Boolean variables. Boolean algebra is a mathematical style dealing in logic. A fundamental rule relating to Boolean variables is called Boolean theorems.

Boolean theorems

1. Cumulative laws
   i. A+B=B+A **ii**. AB=BA

2. Associative laws
   i. (A+B)+C=A(B+C)=A+B+C **ii.** A(BC)=A(BC)=ABC

3. Distributive laws
   i. A(B+C)=AB+AC **ii.** A+BC=(A+B)(A+C)

   - This state that an expression can be expanded by multiplying term by term just like ordinary algebra. It indicates thus we can factor an expression

   i.e – AB'C+A'B'C'=B'(AC+A'C') – Common factor is B'

   - Simplifying by distributive law

   Y=AB'C+AB'D'=AB'(D+D')=AB' – since D+D'=1+0=1 by distributive law

4. Identity law
   i. A+A=A ii. AA=A

5. Negative law
   i. A'=A' ii. A''=A

6. Redundancy laws
   i. A+AB=A(1+B)=A(1)=1 N/b 1+n=1 where n=any num/char
   ii. A(A+B)=AA+AB=A+AB=A
   iii. 0+A=A
   iv. 0A=0
   v. 1+A=1
   vi. 1A=A
   vii. A'+A=1
   viii. A'A=0
   ix. A+AB'=A+B
   x. A(A'+B)=AB

EXAMPLE

Z=(A'+B)(A+B)=AA'+A'B+AB+BB=0+A'B+AB+B=B(A'+A+1)=B(1+1)=B

<u>Proves</u>

i. AC+ABC=AC
   Let y=AC+ABC

   =AC(1+B)=AC since 1+n=1

ii. (A+B)(A+C)=A+BC
   Let y= (A+B)(A+C)

   =A(A+C)+B(A+C)=AA+AC+AB+CB=A+AC+AB+CB

= A(1+B)+AC+BC=A+AC+BC=A(1+C)+CB=A+BC

iii. A+A’B=A+B
Let y= A+A’B

= A.1+A’B=A(1+B)+A’B=A.1+AB+A’B=A+AB+A’B

= A+B(A+A’)=A+B

iv. (A+B)(A+B’)(A’+C)=AC
Let y=(A+B)(A+B’)(A’+C)

= (AA+AB’+BA+BB’)A’+C=(A+AB+AB’)(A’+C)

= [A(1+B)+AB’](A’+C)

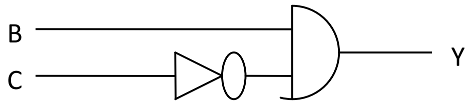
= (A+AB’)(A’+C)=A(1+B’)(A’+C)=A.1(A’+C)=A(A’+C)=AA’+AC=AC

v. Simplify the expression and show minimum gate implementation
y=ABC’D’+A’BC’D’+BC’D

Since A+A’=1 and A.1=A

=BC’D’ (A+A’) +BC’D=BC’D’.1+BC’D

= BC’D’+BC’D=BC’ (D+D’) =BC’.1=BC’



7. Dedmorgans theorems
The theorems are useful in simplifying expressions in which a product or sum of variables is complimented or inverted.

The two theorems are

a) (A+B)’=A’B’
When the OR sum of two variables (A+B) is complimented, this is same as if the 2 variable’s compliments were ANDed.

i.e. – compliment of an OR sum is AND product of the compliment.

b) (AB)’=A’+B’

Compliment of an AND product is equal to OR sum of its compliment

Karnaugh maps (K-maps)

K-maps/ vetch diagram is a method to simplify Boolean expressions. The maps reduce the need for extensive calculations by taking advantage of human pattern-recognition capability.

In K—map, the Boolean variables are transferred (generally from a truth table) and ordered according to the principles of gray code in which only one variable changes in between squares.

Once the table is generated and the output possibilities transcribed, the data is arranged into the largest possible groups containing 2^n cells (n=0, 1, 2, 3...) and the minterms generated through the axiom laws of Boolean algebra

Note

A minterm is a product (AND) of all variables in the function, in directs or complemented form. A minterm has the property that it is equal to 1 on exactly one row of the truth table.

A maxterm is a sum (OR) of all the variables in the function, in direct or complemented form. A maxterm has the property that it is equal to 0 on exactly one row of the truth table.

Don't care conditions are represented by X in the K-Map table. A don't-care term for a function is an input-sequence (a series of bits) for which the function output does not matter (0,1).

AB CD	**00** **(A'B')**	**01** **(A'B)**	**11** **(AB)**	**10** **(AB')**
00 **(C'D')**	M0	M4	M12	M8
01 **(C'D)**	M1	M5	M13	M9
11 **(CD)**	M3	M7	M15	M11
10 **(CD')**	M2	M6	M14	M10

Procedure

K-map method may theoretically be applied to simplify any Boolean expression through works well with <=6 variable.

- Each variable contributes two possibilities. The initial value and its inverse.

The variables are arranged in gray code in which only one variable changes between two adjacent grid boxes.

- Once the variables have been defined, the output possibilities are transcribed according to the grid location provided by the variables. Thus for every possibility of a Boolean input or variable the output possibility is defined.

When the K-map has been completed, to derive a minimized function the one's or desired outputs are grouped into the largest possible rectangular groups in which the num of grid boxes (output possibilities) in the groups must be equal to power of two.

- Don't care**(**s**)** possibilities **(**generally represented by X**)** are grouped only if the group created is larger than the group with minterms.

The boxes can be used more than once if they produce the least number of groups and each desired output must be contained within at least one grouping.

- The groups generated are then converted to a Boolean expression by locating and transcribing the variable possibility attributed to the box, and by the axiom laws of Boolean algebra – in which, if the initial variable possibility and its inverse are contained within the same group the variable term is removed.

Note
Each group provides a "product" to create a "SOP" in the Boolean expression. To determine the inverse of the K-map, the 0's are grouped instead of the 1's. the two expressions are non-complementary.
Each square in a K-map corresponds to a minterm and maxterm in the venn diagram.

Example
Following is an unspecified Boolean algebra function with Boolean variables ABC and D and their inverses
They can be represented in two different ways

- F(A,B,C,D)=∑(6,8,9,10,11,12,13,14,15)
- F(A,B,C,D)=A'BCD'+ AB'C'D'+ AB'C'D+ AB'CD'+ AB'CD+ ABC'D'+ ABCD'

Truth table

Using the defined minterms the table can be created as follow.

M#	ABCD(bin)	ABCD(gray)	F
0	0000	0000	0
1	0001	0001	0
2	0010	0011	0
3	0011	0010	0
4	0100	0110	0
5	0101	0111	0
6	0110	0101	1
7	0111	0100	0
8	1000	1100	1
9	1001	1101	1
10	1010	1111	1
11	1011	1110	1
12	1100	1010	1
13	1101	1011	1
14	1110	1001	1
15	1111	1000	1

K-MAP

The input variable can be combined in 16 different ways, so the K-map has 16 positions and thus is arranged in a 4×4grid.

AB CD	00	01	11	10
00	M0	M4	M12	M8
01	M1	M5	M13	M9
11	M3	M7	M15	M11
10	M3	M6	M14	M10

AB CD	00	01	11	10
00	0	0	1	1
01	0	0	1	1
11	0	0	1	1
10	0	1	1	1

The bin digits in the map rep the function output for any given combination of inputs.

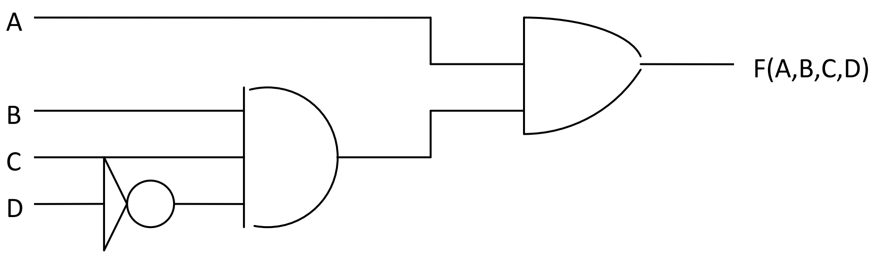
After K-map is constructed, now find the minimal terms to use in the final expression.

AB CD	00	01	11	10
00	0	0	1	1
01	0	0	1	1
11	0	0	1	1
10	0	1	1	1

Therefore the expression is: **F(A,B,C,D)=A+CD'B**

<u>Note</u>

- Encircled groups may overlap.
- The grid is toroidally connected which means the grouping may wrap around edges.



Sum of products (SOP) – Summation of the minterm multiplicities

- It uses the minterms or variables that are high

F(A,B,C,D)=ABC+CD’B

Products of Sums (POS) – Multiplication of the maxterm summations

- It uses the maxterms or variables that are low

F(A,B,C,D)= (A+B)(A+C)(B’+C’+D’)

DON’T CARES

The variables with a don’t care may assume minterms or maxterms so long as they produce the most simple circuit.

F(W,X,Y,Z)=YZ+XY+XZ - SOP

F(W,X,Y,Z)=(X+Z)(X+Y)(X+Z) – POS

WX **YZ**	**00**	**01**	**11**	**10**
00	0	0	X	0
01	0	1	X	0
11	1	1	X	X
10	0	1	x	X

CHAPTER 4: DISCRETE COUNTING

The Basic Rules/Principles of Counting

The Inclusion-Exclusion and the Pigeonhole Principles are the most fundamental combinatorial techniques. There are two additional rules which are basic to most elementary counting. One is known as the *Sum Rule* (or *Disjunctive Rule*), the other is called *Product Rule* (or *Sequential Rule*.)

The Rules of Sum and Product

The **Rule of Sum** and **Rule of Product** are used to decompose difficult counting problems into simple problems.

- **The Rule of Sum** − If a sequence of tasks $T_1, T_2, \ldots, T_m$ can be done in $w_1, w_2, \ldots w_m$ ways respectively (the condition is that no tasks can be performed simultaneously), then the number of ways to do one of these tasks is $w_1 + w_2 + \cdots + w_m$. If we consider two tasks A and B which are disjoint (i.e. $A \cap B = \emptyset$), then mathematically $|A \cup B| = |A| + |B|$
- **The Rule of Product** − If a sequence of tasks $T_1, T_2, \ldots, T_m$ can be done in $w_1, w_2, \ldots w_m$ ways respectively and every task arrives after the occurrence of the previous task, then there are $w_1 \times w_2 \times \cdots \times w_m$ ways to perform the tasks. Mathematically, if a task B arrives after a task A, then $|A \times B| = |A| \times |B|$

Example

Question − A boy lives at X and wants to go to School at Z. From his home X he has to first reach Y and then Y to Z. He may go X to Y by either 3 bus routes or 2 train routes. From there, he can either choose 4 bus routes or 5 train routes to reach Z. How many ways are there to go from X to Z?

Solution – From X to Y, he can go in $3+2=5$ ways (Rule of Sum). Thereafter, he can go Y to Z in $4+5=9$ ways (Rule of Sum). Hence from X to Z he can go in $5 \times 9 = 45$ ways (Rule of Product).

Pigeonhole Principle

In 1834, German mathematician, Peter Gustav Lejeune Dirichlet, stated a principle which he called the drawer principle. Now, it is known as the pigeonhole principle.

Pigeonhole Principle states that if there are fewer pigeon holes than total number of pigeons and each pigeon is put in a pigeon hole, then there must be at least one pigeon hole with more than one pigeon. If n pigeons are put into m pigeonholes where n > m, there's a hole with more than one pigeon.

Examples

- Ten men are in a room and they are taking part in handshakes. If each person shakes hands at least once and no man shakes the same man's hand more than once then two men took part in the same number of handshakes.
- There must be at least two people in a class of 30 whose names start with the same alphabet.

The Inclusion-Exclusion principle

The **Inclusion-exclusion principle** computes the cardinal number of the union of multiple non-disjoint sets. For two sets A and B, the principle states −

$$|A\cup B|=|A|+|B|-|A\cap B|$$

For three sets A, B and C, the principle states −

$$|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$

The generalized formula -

$$|\bigcup_{ni=1}A_i|=\sum_{1\le i<j<k\le n}|A_i\cap A_j|+\sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k|-\cdots+(-1)_{\pi-1}|A_1\cap\cdots\cap A_2|$$

Problem 1
How many integers from 1 to 50 are multiples of 2 or 3 but not both?

Solution
From 1 to 100, there are $50/2=25$ numbers which are multiples of 2.

There are $50/3=16$ numbers which are multiples of 3.

There are $50/6=8$ numbers which are multiples of both 2 and 3.

So, $|A|=25$, $|B|=16$ and $|A\cap B|=8$.

$$|A\cup B|=|A|+|B|-|A\cap B|=25+16-8=33$$

Problem 2
In a group of 50 students 24 like cold drinks and 36 like hot drinks and each student likes at least one of the two drinks. How many like both coffee and tea?
Solution
Let X be the set of students who like cold drinks and Y be the set of people who like hot drinks.

So, $|X\cup Y|=50$, $|X|=24$, $|Y|=36$

$|X \cap Y|=|X|+|Y|-|X \cup Y|=24+36-50=60-50=10$

Counting Techniques

Permutations

A **permutation** is an arrangement of some elements in which order matters. In other words a Permutation is an ordered Combination of elements.

Examples

- From a set S ={x, y, z} by taking two at a time, all permutations are − xy, yx, xz, zx, yz, zy
- We have to form a permutation of three digit numbers from a set of numbers $S=\{1,2,3\}$. Different three digit numbers will be formed when we arrange the digits. The permutation will be = 123, 132, 213, 231, 312, 321

Number of Permutations

The number of permutations of 'n' different things taken 'r' at a time is denoted by nP_r

$$nP_r=n!(n-r)!$$

where $n!=1.2.3.\ldots.(n-1).n$

Proof − Let there be 'n' different elements.

There are n numbers of ways to fill up the first place. After filling the first place (n-1) number of elements is left. Hence, there are (n-1) ways to fill up the second place. After filling the first and second place, (n-2) number of elements is left. Hence, there are (n-2) ways to fill up the third place. We can now generalize the number of ways to fill up r-th place as [n – (r–1)] = n–r+1

So, the total no. of ways to fill up from first place up to r-th-place −

$$nP_r=n(n-1)(n-2).....(n-r+1)$$

$$=[n(n-1)(n-2)...(n-r+1)][(n-r)(n-r-1)\ldots 3.2.1]/[(n-r)(n-r-1)\ldots 3.2.1]$$

Hence,

$$nP_r=n!/(n-r)!$$

Some important formulas of permutation

- If there are *n* elements of which a_1 are alike of some kind, a_2 are alike of another kind; a_3 are alike of third kind and so on and a_r are of r_{th} kind, where $(a_1+a_2+...a_r)=n$.

  Then, number of permutations of these n objects is = $n!/[(a_1!(a_2!)\ldots(a_r!)]$.

- Number of permutations of n distinct elements taking n elements at a time = $nP_n=n!$
- The number of permutations of n dissimilar elements taking r elements at a time, when x particular things always occupy definite places = $n-x_{p_{r-x}}$
  - The number of permutations of n dissimilar elements when r specified things always come together is − $r!(n-r+1)!$
  - The number of permutations of n dissimilar elements when r specified things never come together is − $n!-[r!(n-r+1)!]$
  - The number of circular permutations of n different elements taken x elements at time = $_np_x/x$
  - The number of circular permutations of n different things = $_np_n/n$

Some Problems

Problem 1 − From a bunch of 6 different cards, how many ways we can permute it?

Solution − As we are taking 6 cards at a time from a deck of 6 cards, the permutation will be $_6P_6=6!=720$

Problem 2 − In how many ways can the letters of the word 'READER' be arranged?

Solution − There are 6 letters word (2 E, 1 A, 1D and 2R.) in the word 'READER'.

The permutation will be $=6!/[(2!)(1!)(1!)(2!)]=180.$

Problem 3 − In how ways can the letters of the word 'ORANGE' be arranged so that the consonants occupy only the even positions?

Solution − There are 3 vowels and 3 consonants in the word 'ORANGE'. Number of ways of arranging the consonants among themselves $=_3P_3=3!=6$. The remaining 3 vacant places will be filled up by 3 vowels in $_3P_3=3!=6$ ways. Hence, the total number of permutation is $6\times6=36$

Combinations

A **combination** is selection of some given elements in which order does not matter.

The number of all combinations of n things, taken r at a time is −

$${}_nC_r=n!r!(n-r)!$$

Problem 1

Find the number of subsets of the set $\{1,2,3,4,5,6\}$having 3 elements.

Solution

The cardinality of the set is 6 and we have to choose 3 elements from the set. Here, the ordering does not matter. Hence, the number of subsets will be ${}_6C_3=20$.

Problem 2

There are 6 men and 5 women in a room. In how many ways we can choose 3 men and 2 women from the room?

Solution

The number of ways to choose 3 men from 6 men is ${}_6C_3$ and the number of ways to choose 2 women from 5 women is ${}_5C_2$

Hence, the total number of ways is − ${}_6C_3\times{}_5C_2=20\times10=200$

Problem 3

How many ways can you choose 3 distinct groups of 3 students from total 9 students?

Solution

Let us number the groups as 1, 2 and 3

For choosing 3 students for 1st group, the number of ways − ${}_9C_3$

The number of ways for choosing 3 students for 2nd group after choosing 1st group − ${}_6C_3$

The number of ways for choosing 3 students for 3rd group after choosing 1st and 2nd group − ${}_3C_3$

Hence, the total number of ways $=9C3\times6C3\times3C3=84\times20\times1=1680$

Pascal's Identity

Pascal's identity, first derived by Blaise Pascal in 19th century, states that the number of ways to choose k elements from n elements is equal to the summation of number of ways to choose (k-1) elements from (n-1) elements and the number of ways to choose elements from n-1 elements.

Mathematically, for any positive integers k and n: $nCk=n-1Ck-1+n-1Ck$

Proof –

$$n-1Ck-1+n-1Ck$$

$=(n-1)!(k-1)!(n-k)!+(n-1)!k!(n-k-1)!$

$=(n-1)!(kk!(n-k)!+n-kk!(n-k)!)$

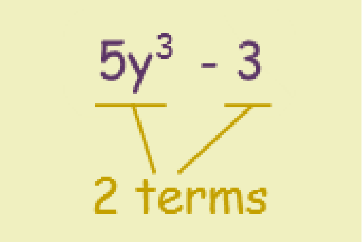
$=(n-1)!nk!(n-k)!$

$=n!k!(n-k)!$

$=nCk$

Binomial Theorem

A **binomial** is a polynomial with two terms



example of a binomial

What happens when we multiply a binomial by itself ... many times?

Example: a+b

a+b is a binomial (the two terms are **a** and **b**)

Let us multiply **a+b** by itself using Polynomial Multiplication :

$$(a+b)(a+b) = \mathbf{a^2 + 2ab + b^2}$$

Now take that result and multiply by **a+b** again:

$$(a^2 + 2ab + b^2)(a+b) = \mathbf{a^3 + 3a^2b + 3ab^2 + b^3}$$

And again:

$$(a^3 + 3a^2b + 3ab^2 + b^3)(a+b) = \mathbf{a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4}$$

The calculations get longer and longer as we go, but there is some kind of **pattern** developing.

That pattern is summed up by the **Binomial Theorem**:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

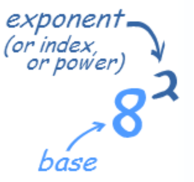
The Binomial Theorem

Don't worry ... it will all be explained!

And you will learn lots of cool math symbols along the way.

Exponents

First, a quick summary of Exponents.



An exponent says **how many times** to use something in a multiplication.

Example: $8^2 = 8 \times 8 = 64$

An exponent of **1** means just to have it appear once, so we get the original value:

Example: $8^1 = 8$

An exponent of **0** means not to use it at all, and we have only 1:

Example: $8^0 = 1$

Exponents of (a+b)

Now on to the binomial.

We will use the simple binomial a+b, but it could be any binomial.

Let us start with an exponent of **0** and build upwards.

Exponent of 0: When an exponent is 0, we get **1**:

$$(a+b)^0 = 1$$

Exponent of 1: When the exponent is 1, we get the original value, unchanged:

$$(a+b)^1 = a+b$$

Exponent of 2: An exponent of 2 means to multiply by itself.

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

Exponent of 3: For an exponent of 3 just multiply again:

$$(a+b)^3 = (a^2 + 2ab + b^2)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

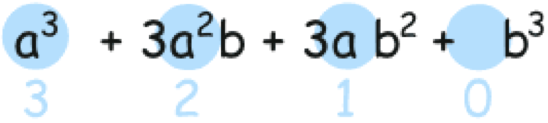
We have enough now to start talking about the pattern.

The Pattern

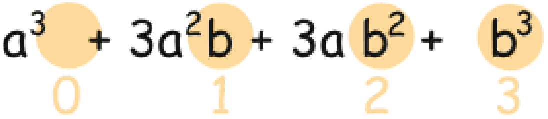
In the last result we got:

$$a^3 + 3a^2b + 3ab^2 + b^3$$

Now, notice the exponents of a. They start at 3 and go down: 3, 2, 1, 0:



Likewise the exponents of b go upwards: 0, 1, 2, 3:



If we number the terms 0 to *n*, we get this:

k=0	k=1	k=2	k=3
a^3	a^2	a	1
1	b	b^2	b^3

Which can be brought together into this:

$$a^{n-k}b^k$$

How about an example to see how it works:

Example: When the exponent, *n*, is 3.

The terms are:

k=0:	**k=1:**	**k=2:**	**k=3:**
$a^{n-k}b^k$ $= a^{3-0}b^0$ $= \mathbf{a^3}$	$a^{n-k}b^k$ $= a^{3-1}b^1$ $= \mathbf{a^2b}$	$a^{n-k}b^k$ $= a^{3-2}b^2$ $= \mathbf{ab^2}$	$a^{n-k}b^k$ $= a^{3-3}b^3$ $= \mathbf{b^3}$

It works like magic!

Coefficients

So far we have:	$a^3 + a^2b + ab^2 + b^3$
But we **really** need:	$a^3 + 3a^2b + 3ab^2 + b^3$

We are **missing the numbers** (which are called *coefficients*).

Let's look at **all the results** we got before, from $(a+b)^0$ up to $(a+b)^3$:

$$1$$

$$a + b$$

$$a^2 + 2ab + b^2$$

$$a^3 + 3a^2b + 3ab^2 + b^3$$

And now look at **just the coefficients** (with a "1" where a coefficient wasn't shown):

$$1$$

$$1a + 1b$$

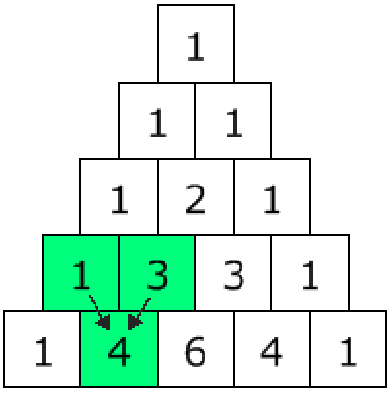
$$1a^2 + 2ab + 1b^2$$

$$1a^3 + 3a^2b + 3ab^2 + 1b^3$$

They actually make Pascal's Triangle!

Each number is just the two numbers above it added together (except for the edges, which are all "1")

(Here I have highlighted that **1+3 = 4)**



Armed with this information let us try something new ... an **exponent of 4**:

a exponents go 4,3,2,1,0:		a^4	+	a^3	+	a^2	+	a	+	1	
b exponents go 0,1,2,3,4:		a^4	+	a^3b	+	a^2b^2	+	ab^3	+	b^4	
coefficients go 1,4,6,4,1:		a^4	+	$4a^3b$	+	$6a^2b^2$	+	$4ab^3$	+	b^4	✓

And that is the correct answer (compare to the top of the page).

We have success!

We can now use that pattern for exponents of 5, 6, 7, ... 50, ... 112, ... you name it!

That pattern is the essence of the Binomial Theorem.

Now you can take a break.

When you come back see if you can work out $\mathbf{(a+b)^5}$ yourself.

Answer (hover over): $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Formula

Our last step is to write it all as a formula.

But hang on, how do we write a formula for **"find the coefficient from Pascal's Triangle"** ... ?

Well, there **is** such a formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It is commonly called "n choose k" because it is how many ways to choose k elements from a set of n.

You can read more at Combinations and Permutations

The "!" means "factorial", for example 4! = 4×3×2×1 = 24

And it matches to Pascal's Triangle like this:

(Note how the top row is row zero and also the leftmost column is zero!)

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

Example: Row 4, term 2 in Pascal's Triangle is "6".

Let's see if the formula works:

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

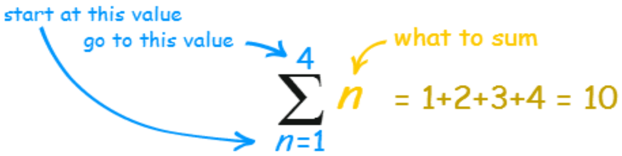
Yes, it works! Try another value for yourself.

Putting It All Together

The last step is to put all the terms together into **one formula**.

But we are adding lots of terms together ... can that be done using one formula?

Yes! The handy Sigma Notation allows us to sum up as many terms as we want:



Sigma Notation

Now it can all go into one formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The Binomial Theorem

Use It

OK ... it won't make much sense without an example.

So let's try using it for *n* = 3 :

$$\begin{aligned}(a+b)^3 &= \sum_{k=0}^{3} \binom{3}{k} a^{3-k} b^k \\ &= \binom{3}{0} a^{3-0} b^0 + \binom{3}{1} a^{3-1} b^1 + \binom{3}{2} a^{3-2} b^2 + \binom{3}{3} a^{3-3} b^3 \\ &= 1 \cdot a^3 b^0 + 3 \cdot a^2 b^1 + 3 \cdot a^1 b^2 + 1 \cdot a^0 b^3 \\ &= a^3 + 3a^2 b + 3ab^2 + b^3\end{aligned}$$

BUT ... it is usually **much easier** just to remember the **patterns**:

- The first term's exponents start at **n and go down**
- The second term's exponents start at **0 and go up**
- Coefficients are from Pascal's Triangle, or by calculation using **n!/(k!(n-k)!)**

Like this:

Example: What is $(y+5)^4$

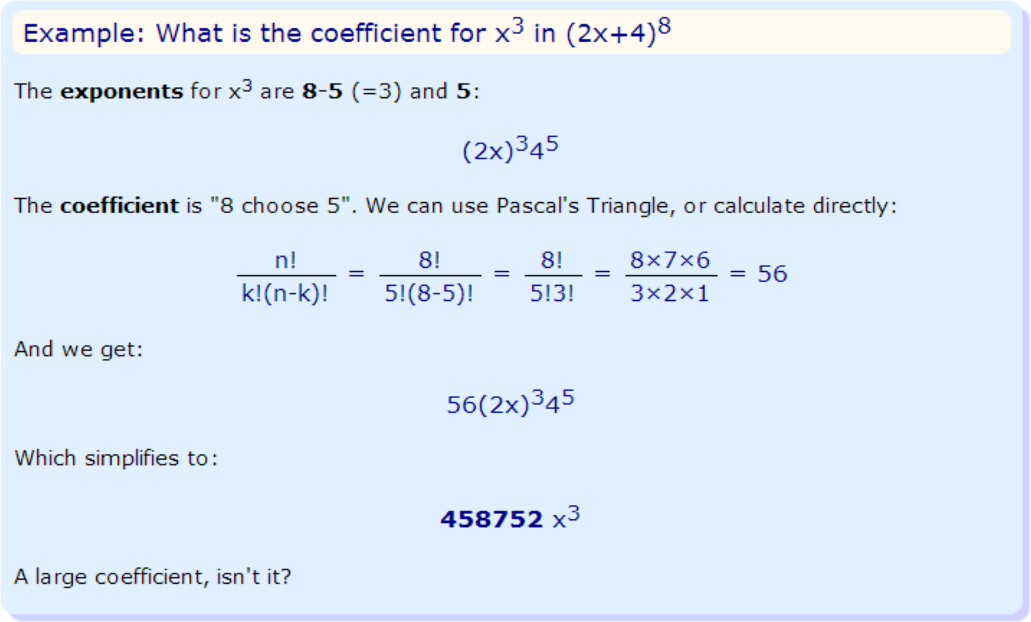
Start with exponents:	$\mathbf{y^4 5^0}$	$\mathbf{y^3 5^1}$	$\mathbf{y^2 5^2}$	$\mathbf{y^1 5^3}$	$\mathbf{y^0 5^4}$
Include Coefficients:	$\mathbf{1}y^4 5^0$	$\mathbf{4}y^3 5^1$	$\mathbf{6}y^2 5^2$	$\mathbf{4}y^1 5^3$	$\mathbf{1}y^0 5^4$

Then write down the answer (including all calculations, such as 4×5, 6×5^2, etc):

$$(y+5)^4 = y^4 + 20y^3 + 150y^2 + 500y + 625$$

We may also want to calculate just one term:

Example: What is the coefficient for x^3 in $(2x+4)^8$



Example: What is the coefficient for x^3 in $(2x+4)^8$

The **exponents** for x^3 are **8-5** (=3) and **5**:

$$(2x)^3 4^5$$

The **coefficient** is "8 choose 5". We can use Pascal's Triangle, or calculate directly:

$$\frac{n!}{k!(n-k)!} = \frac{8!}{5!(8-5)!} = \frac{8!}{5!3!} = \frac{8\times7\times6}{3\times2\times1} = 56$$

And we get:

$$56(2x)^3 4^5$$

Which simplifies to:

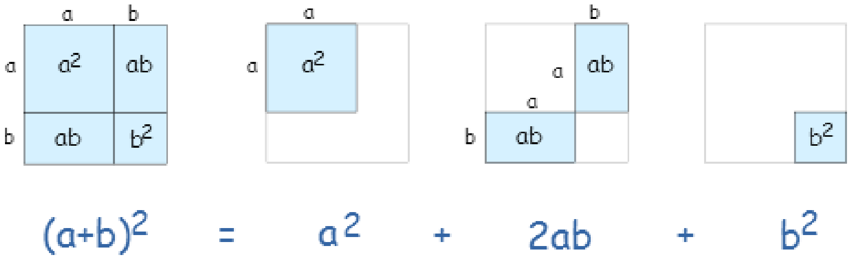
$$\mathbf{458752}\ x^3$$

A large coefficient, isn't it?

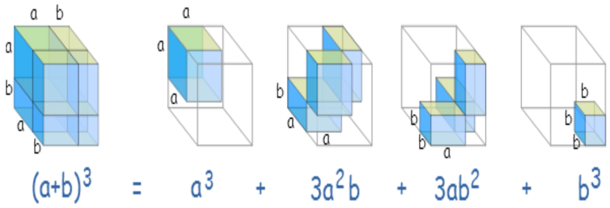
Geometry

Want to see the Binomial Theorem using Geometry?

In 2 dimensions, $(a+b)^2 = a^2 + 2ab + b^2$



In 3 dimensions, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$



In 4 dimensions, $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

(Sorry, I am not good at drawing in 4 dimensions!)

Advanced Example

And one last, most amazing example:

Example: A formula for e (Euler's Number)

We can use the Binomial Theorem to calculate e (Euler's number).

e = **2.718281828459045...** (the digits go on forever without repeating)

It can be calculated using:

$$(1 + 1/n)^n$$

(It gets more accurate the higher the value of **n**)

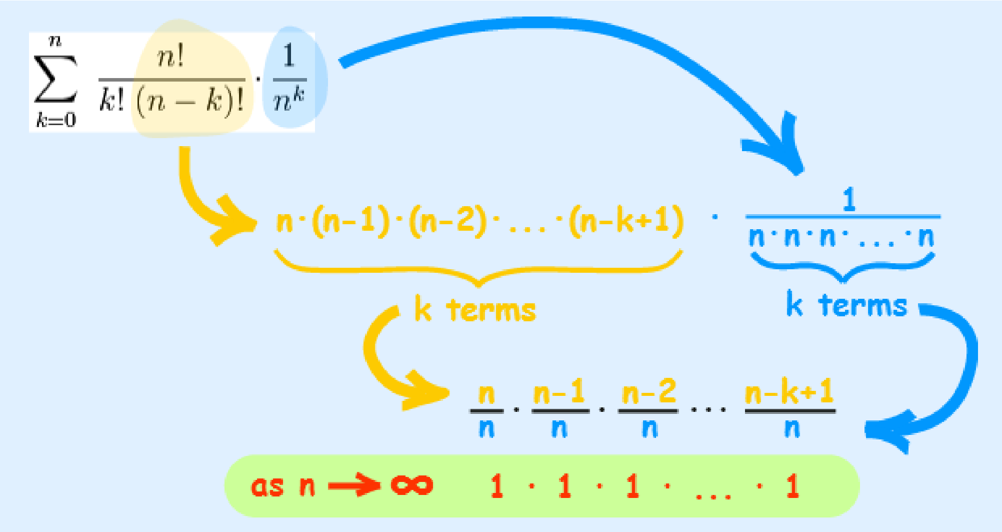
That formula is a **binomial**, right? So let's use the Binomial Theorem:

$$(1+\tfrac{1}{n})^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (\tfrac{1}{n})^k$$

First, we can drop 1^{n-k} as it is always equal to 1:

$$= \sum_{k=0}^{n} \binom{n}{k} (\tfrac{1}{n})^k$$

And, quite magically, most of what is left goes to **1** as n goes to infinity:



Which just leaves:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$
$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots$$

With just those first few terms we get e ≈ 2.7083...

Try calculating more terms for a better approximation!

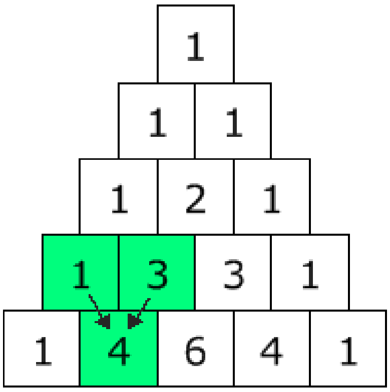
Pascal's Triangle

One of the most interesting Number Patterns is Pascal's Triangle (named after *Blaise Pascal*, a famous French Mathematician and Philosopher).

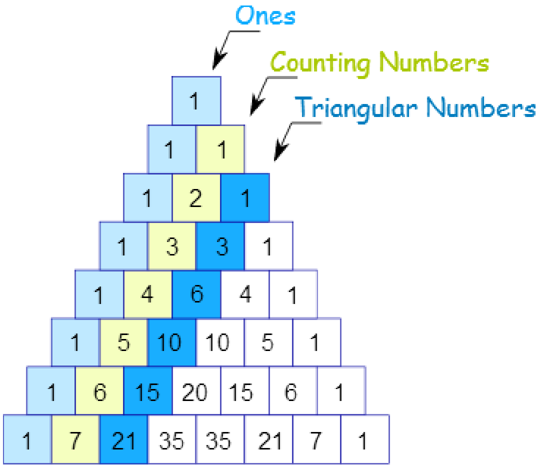
To build the triangle, start with "1" at the top, then continue placing numbers below it in a triangular pattern.

Each number is the numbers directly above it added together.

(Here I have highlighted that **1+3 = 4)**



Patterns Within the Triangle



Diagonals

The first diagonal is, of course, just "1"s

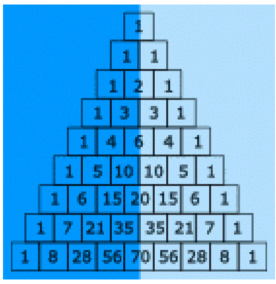
The next diagonal has the Counting Numbers (1,2,3, etc).

The third diagonal has the triangular numbers

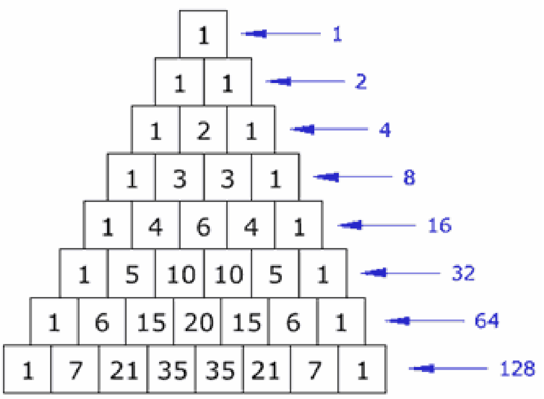
(The fourth diagonal, not highlighted, has the tetrahedral numbers.)

Symmetrical

The triangle is also symmetrical. The numbers on the left side have identical matching numbers on the right side, like a mirror image.



Horizontal Sums



What do you notice about the horizontal sums?

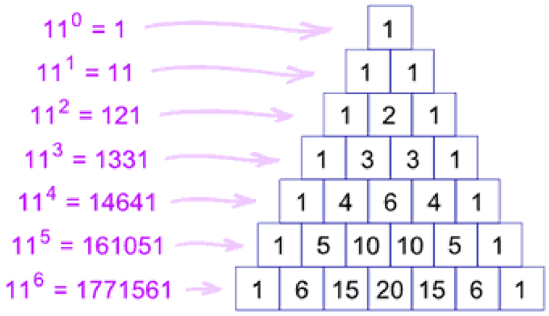
Is there a pattern?

They **double** each time (powers of 2).

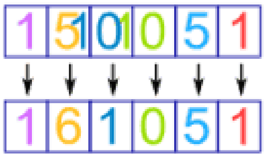
Exponents of 11

Each line is also the powers (exponents) of 11:

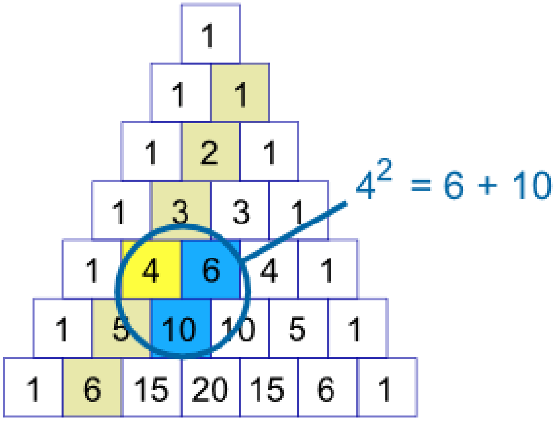
- 11^0=1 (the first line is just a "1")
- 11^1=11 (the second line is "1" and "1")
- 11^2=121 (the third line is "1", "2", "1")
- etc!



But what happens with 11^5? Simple! The digits just overlap, like this:



The same thing happens with 11^6 etc.



Squares

For the second diagonal, the square of a number is equal to the sum of the numbers next to it and below both of those.

Examples:

- $3^2 = 3 + 6 = 9$,
- $4^2 = 6 + 10 = 16$,
- $5^2 = 10 + 15 = 25$,
- ...

There is a good reason, too ... can you think of it? (Hint: 4^2=6+10, 6=3+2+1, and 10=4+3+2+1)

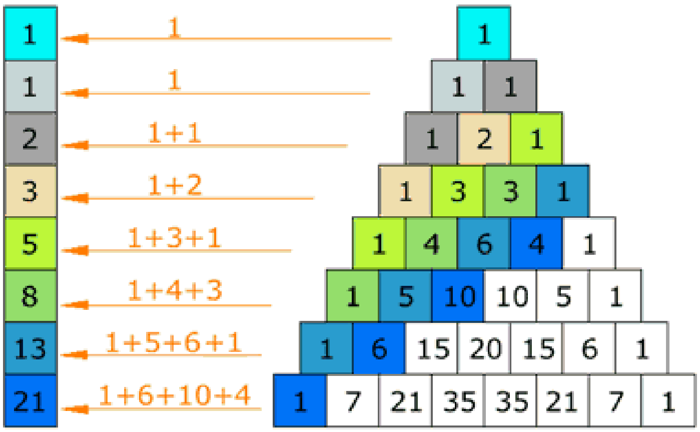
Fibonacci Sequence

Try this: make a pattern by going up and then along, then add up the values

(The Fibonacci Sequence starts "0, 1" and then continues by adding the two previous numbers, for example 3+5=8, then 5+8=13, etc)

The Fibonacci Sequence is the series of numbers:

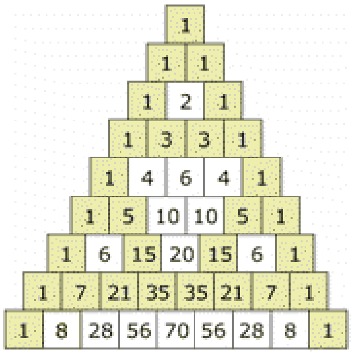
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...



The next number is found by adding up the two numbers before it.

- The 2 is found by adding the two numbers before it (1+1)
- The 3 is found by adding the two numbers before it (1+2),
- And the 5 is (2+3),
- and so on!

Example: the next number in the sequence above is 21+34 = **55**



Odds and Evens

If you color the Odd and Even numbers, you end up with a pattern the same as the Sierpinski Triangle

Here is how you can create one:

1. Start with a triangle.
2. Shrink the triangle to half height, and put a copy in each of the three corners
3. Repeat step 2 for the smaller triangles, again and again, for ever!

CHAPTER 5: PROBABILITY

Introduction to probability

Probability is the science of studying the outcomes of *random phenomena*.

A phenomenon is called *random* if individual outcomes are uncertain but the long-term pattern of many outcomes is predictable.

To determine the probability of a specific outcome of a probability experiment (random phenomenon) the experiment is repeated a large number of times and we look at the ratio of times that the outcome occurred.

- *Example:* The outcome of the experiment of tossing a coin is a random phenomenon. The probability that the outcome is "Heads" can be found by tossing the coin a large number of times.

Probability Models

A **probability model** is a mathematical representation of a random phenomenon. It is defined by its sample space, events within the sample space, and **probabilities** associated with each event. The sample space S for a **probability model** is the set of all possible outcomes.

Terminology:

For any probability experiment (or a random phenomenon)

- ✓ The collection of all the possible outcomes of the experiment is called the **Sample Space**.
- ✓ An **event** is any sub-collection of outcomes in the sample space.
- ✓ A **simple event** is any single outcome.
- ✓ The **complement** of an event *A*, denoted $\bar{A}$, is the set of all outcomes not in *A*.
- ✓ A **probability model** (or a **probability distribution**) is a process of assigning to the outcomes and events in a sample space a value that represents the probability for that outcome to occur. If x is an outcome, the probability of x will be denoted by $P(x)$. If E is an event, the probability of E will be denoted by $P(E)$.
- ✓ The assignment in a probability model must satisfy the following rules:
  - For any event or outcome, **the probability is a number between 0 and 1**.
  - **The probability of the sample space is 1.**
  - **If two events A and B do are disjoint,** i.e. have no outcome in common, **then the probability P(A or B)** that one or the other occurs **is the sum** $P(A) + P(B)$ of the individual probabilities of the events.
  - The probability of an impossible (empty) event is 0.

For a finite (discrete) probability space, these rules guarantee that **the probabilities of the individual events must add up to 1.**

Example:

Suppose we have the following proportions for the marital status of a people in this country between the ages of 30 and 40.

marital status	Never married	Married	Widowed	Divorced
probability	.273	.603	.004	

A random experiment is drawing a person aged 30 to 40 at random. What is the sample space for this experiment?

1) What is the event: "The person is not currently married"

2) What is the complement of this event?

3) If you draw a person aged 30 to 40 at random, what is the probability that:

   a) the person is divorced?

   b) The person is not currently married?

   c) The person has never married, or is widowed?

Equally likely outcomes

If a probability experiment has N outcomes, and all are equally likely to occur, the probability of an individual outcome is 1/N.

The probability of an event A is

P(A) = (The number of outcomes that satisfy A) **)** / (Total number of outcomes)

Examples:

1) The experiment of rolling a fair die has six equally likely outcomes (1, 2, 3, 4, 5, or 6).

   The probability of each of them to occur is 1/6.

   a) Let A be the event "The outcome is a multiple of 3". List the elements in A.

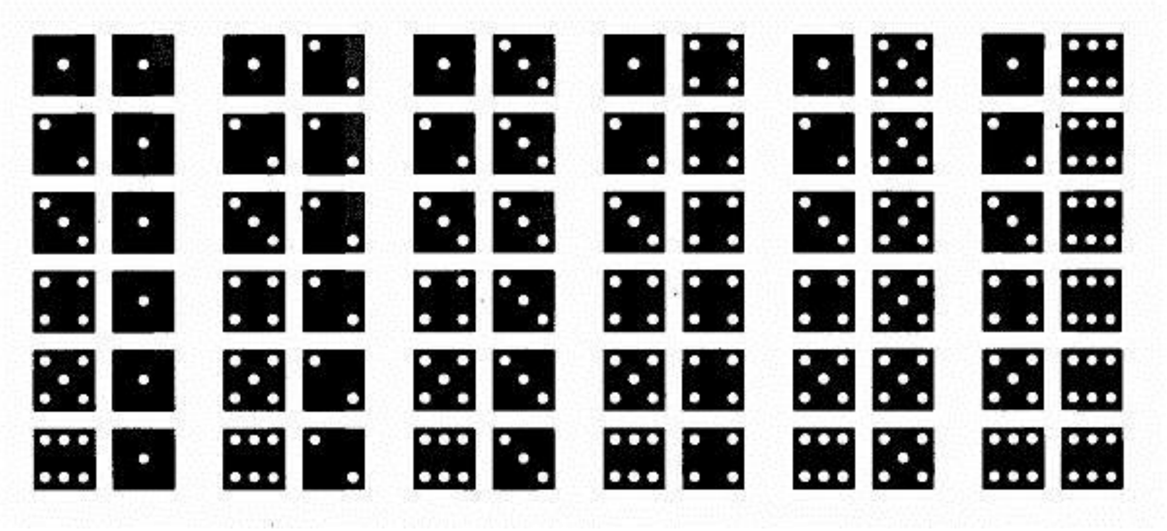
   b) List the elements in $\overline{A}$.

   c) What is $P(A)$?

2) Now consider the experiment of rolling two dice, and recording the number of dots on top,

where the order is important.

a) How many outcomes are there? What is the sample space?



b) What is the probability of rolling a six?

c) In the board game *MONOPOLY* you need to “doubles” to get out of jail. What is the probability of getting out in a single roll?

d) Let *A* be the event that the sum of the number is a multiple of 3. What is *A*?

e) What is *P*(*A*)?

f) Let *E* be the event “ The sum of the two dice is 4". What is *P*(*E*)?

g) What is $\overline{E}$? What is $P(\overline{E})$?

The last two examples illustrate the following rule involving the complement of an event.

- The Complement Rule

$P(\overline{A}) = 1 - P(A)$

Examples:

1) A statistics student who takes the bus to school every day uses relative frequency to find the probabilities that the bus will arrive before a given time. The probability that the bus will come before 8:15 is estimated to be .43. What is the probability that the bus will not come before 8:15?

20 a) A computer assigns a you a PIN at random by assigning 4 symbols, the first is a letter of

the alphabet, the other three are digits from 0 - 9 (with repetition allowed) . Your favorite number is 6. What is the probability of getting a pin with at least one 6?

b) What is the probability of getting a 6 if repeated digits are not allowed?

The Sum Rule

If two events have no elements in common, they follow the rule: $P(AorB) = P(A) + P(B)$

Definition: If two events have no elements in common, they are called **disjoint**

Example

In the example of rolling two die and recording the pair of numbers, let *A* be the event "the sum of the two numbers is a 3", and let *B* be the event "the sum of the two numbers is a 5". List the elements in *A*; list the elements in *B*; and list all the elements in the event " the sum of the two numbers is a 3 or a 5". Find *P*(*A*), *P*(*B*), and the probability that the sum is a 3 or a 5.

But if the events have elements in common this doesn't work.

Example:

In the example of rolling two die and recording the pair of numbers, let *C* be the event "one of the two numbers is a 3", and let *B* be the event "the sum of the two numbers is a 5". List the elements in *C*; list the elements in *B*; and list all the elements in the event " one of the two numbers is a 3 or the sum of the two numbers is a 5". Find *P*(*C*), *P*(*B*), and the probability that one of the numbers is a 3 or the sum is a 5.

This illustrates the sum rule: $P(AorB) = P(A) + P(B) - P(AandB)$, where $P(AandB)$ is the probability that both A and B will happen.

Examples

1) A card is drawn at random from a standard deck of cards. What is the probability it will be a king or a heart?

2) The data for a group of women from a survey is given below:

Status	age 20-34	age 35-49	total
never married	152	51	203
married	226	312	538
widowed	18	69	87
divorced	76	96	172
Total	472	528	1000

If you select a woman at random, what is the probability that she is either married, or over 35?

Random Variables

A random variable gives a value to each outcome in a sample space. A probability model for a random variable assigns a probability value to each variable value.

Examples:

1) In the dice rolling experiment: If we are only interested in the sum of the die, the random variable would be the sum of the two numbers.

2) In the experiment of rolling one die, the random variable could be the number of dots that lands on top.

3) In an experiment of tossing a coin, a random variable could assign the number 1 to *heads* and 0 to *tails*.

A probability model (or probability distribution) satisfies the rules:

- ✓ For any variable, **the probability is a number between 0 and 1**.
- ✓ The probability of the sum of all the variables is 1.

The Mean (Expected Value) of a Probability model

Suppose that the possible values for the random variables are $s_1, s_2, s_3, \ldots, s_k$; and suppose that the probability of the variable s_j is p_j. Then

the *mean* (also called *expected value*) F of the probability distribution is

$$F = s_1 p_1 + s_2 p_2 + \ldots + s_k p_k$$

Example: The following represents a table of grades given to a class for a term paper. An experiment consists of drawing a student at random and recording that student's grade. The random variable is the value of the grade. The probability distribution is given in the table.

Grade	0	1	2	3	4
Probability	0.10	0.15	0.30	0.30	0.15

Find the mean of the distribution.

Example:

In a gambling game, the player is paid $3 if s/he draws a queen, $5 if s/he draws a king, and $10 if s/he draws an ace. Otherwise, the player will pay the casino the bet amount of $3.00.

1. Find the probability model representing the amount of money the player wins (negative number for loss!)

x				
$P(x)$				

2. Find the mean of this model. Does the game favor the casino or the player?

3. What should the amount of the bet be so that the game is fair?

The Law of Large Numbers

As a random phenomenon is repeated a large number of times

< The proportion of trials on which an outcome occurs gets closer and closer to the probability of that outcome.

< The average of the observed values gets closer and closer to *F*.

An American roulette wheel has 38 slots numbered 0, 00, and 1 to 36. The ball is equally likely to rest in any of these slots when the wheel is spun. One way to place a bet is to bet that the ball will rest on an odd number. Sam places a $10 bet that pays out $20 if an odd number comes up.

a. What is expected value for one play, taking into account the $10 cost of each play? On any individual play, will he make the expected value?

b. Sam plays roulette every day for 10 years. What does the law of large numbers tell us about his results?

Probability of Single or Combined Events

SINGLE EVENT **PROBABILITY**. The chance of some event happening; such as flipping a coin and having it land with the head side up, or rolling a "four" on a **single** die, is called **probability**. **Probability** is expressed as a fraction. ... Therefore, the **probability** of a "four" showing on the die after one throw is 1/6.

Multiple Probabilities. The **probability** of 2 things happening: The **probability** of 2 independent things happening either one after the other or together is the **probability** of the first thing happening multiplied by the **probability** of the second thing happening.

Independent and Dependent Events

Events can be placed into two major categories dependent or Independent events.

Independent - When two events are said to be independent of each other, what this means is that the probability that one event occurs in no way affects the probability of the other event occurring. An example of two independent events is as follows; say you rolled a die and flipped a coin. The probability of getting any number face on the die in no way influences the probability of getting a head or a tail on the coin.

Dependent - When two events are said to be dependent, the probability of one event occurring influences the likelihood of the other event.

For example, if you were to draw a two cards from a deck of 52 cards. If on your first draw you had an ace and you put that aside, the probability of drawing an ace on the second draw is greatly changed because you drew an ace the first time. Let's calculate these different probabilities to see what's going on.

There are 4 Aces in a deck of 52 cards

$$\mathrm{P(Ace)} = \frac{\text{number of Aces in a deck of cards}}{\text{number of cards in a deck}}$$

On your first draw, the probability of getting an ace is given by:

$$\mathrm{P(Ace)} = \frac{4}{52} = \frac{1}{13}$$

If we don't return this card into the deck, the probability of drawing an ace on the second pick is given by

$$\mathrm{P(Ace)} = \frac{\text{number of Aces remaining in the deck of cards}}{\text{number of cards remaining in a deck}}$$

$$\mathrm{P(Ace)} = \frac{4-1}{52-1}$$

$$\mathrm{P(Ace)} = \frac{3}{51}$$

As you can clearly see, the above two probabilities are different, so we say that the two events are dependent. The likelihood of the second event depends on what happens in the first event.

Conditional Probability

Conditional probability deals with further defining dependence of events by looking at probability of an event given that some other event first occurs.

Conditional probability is denoted by the following:

$$P(B|A)$$

The above is read as **the probability that B occurs given that A has already occurred**.

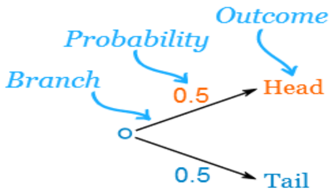
The above is mathematically defined as:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Probability Tree Diagrams

Calculating probabilities can be hard, sometimes we add them, sometimes we multiply them, and often it is hard to figure out what to do ... **tree diagrams to the rescue!**

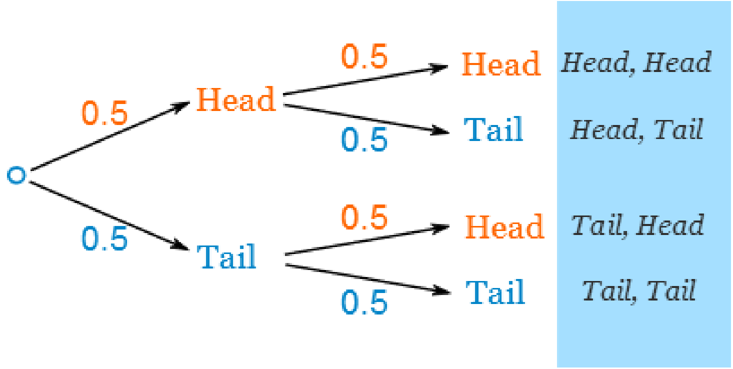
Here is a tree diagram for the toss of a coin:



There are two "branches" (Heads and Tails)

- The probability of each branch is written on the branch
- The outcome is written at the end of the branch

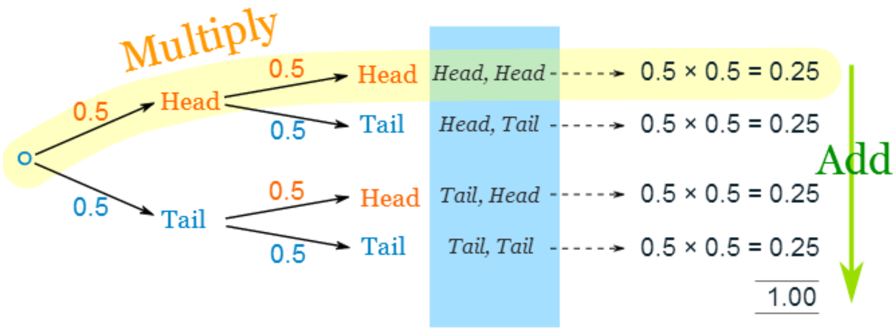
We can extend the tree diagram to two tosses of a coin:



How do we calculate the overall probabilities?

- We **multiply** probabilities **along the branches**

- We **add** probabilities down **columns**



Now we can see such things as:

- The probability of "Head, Head" is 0.5×0.5 = **0.25**
- All probabilities add to **1.0** (which is always a good check)
- The probability of getting at least one Head from two tosses is 0.25+0.25+0.25 = **0.75**
- ... and more

That was a simple example using independent events (each toss of a coin is independent of the previous toss), but tree diagrams are really wonderful for figuring out dependent events (where an event **depends on** what happens in the previous event) like this example:

Example: Soccer Game

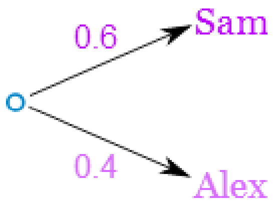
You are off to soccer, and love being the Goalkeeper, but that depends who is the Coach today:

- with Coach Sam the probability of being Goalkeeper is **0.5**
- with Coach Alex the probability of being Goalkeeper is **0.3**

Sam is Coach more often ... about 6 out of every 10 games (a probability of **0.6**).

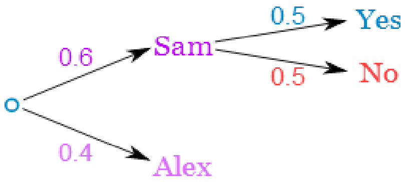
So, what is the probability you will be a Goalkeeper today?

Let's build the tree diagram. First we show the two possible coaches: Sam or Alex:

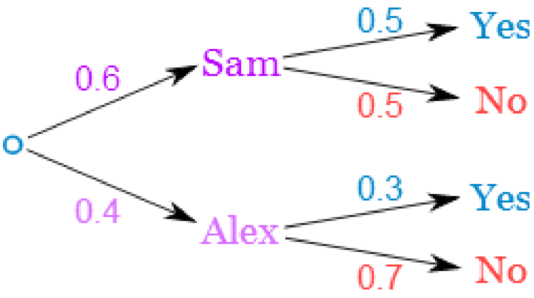


The probability of getting Sam is 0.6, so the probability of Alex must be 0.4 (together the probability is 1)

Now, if you get Sam, there is 0.5 probability of being Goalie (and 0.5 of not being Goalie):

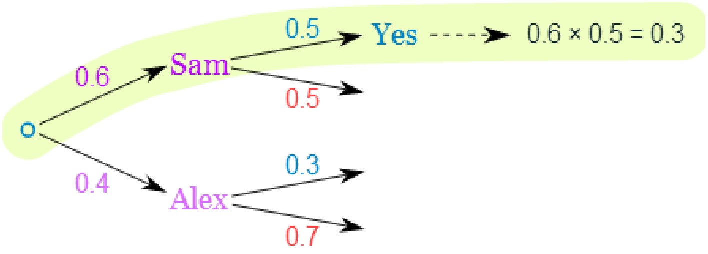


If you get Alex, there is 0.3 probability of being Goalie (and 0.7 not):



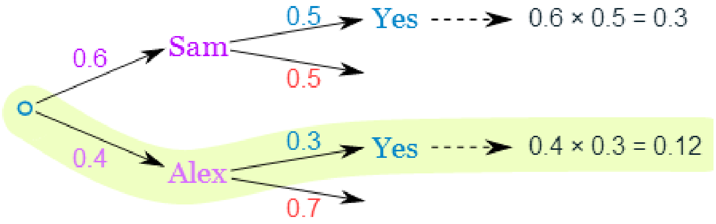
The tree diagram is complete, now let's calculate the overall probabilities. This is done by multiplying each probability along the "branches" of the tree.

Here is how to do it for the "Sam, Yes" branch:



(When we take the 0.6 chance of Sam being coach and include the 0.5 chance that Sam will let you be Goalkeeper we end up with an 0.3 chance.)

But we are not done yet! We haven't included Alex as Coach:



An 0.4 chance of Alex as Coach, followed by an 0.3 chance gives 0.12.

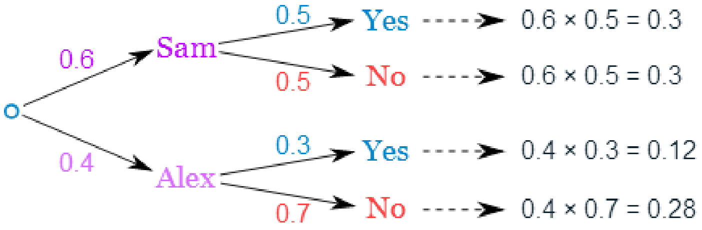
Now we add the column:

0.3 + 0.12 = **0.42 probability** of being a Goalkeeper today

(That is a 42% chance)

Check

One final step: complete the calculations and make sure they add to 1:



$$0.3 + 0.3 + 0.12 + 0.28 = 1$$

Yes, it all adds up.

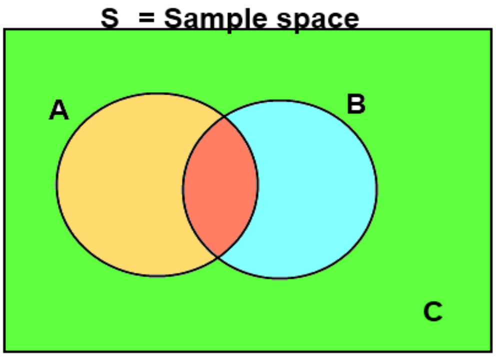
Conclusion

So there you go, when in doubt draw a tree diagram, multiply along the branches and add the columns. Make sure all probabilities add to 1 and you are good to go.

Set Theory in Probability

A sample space is defined as a universal set of all possible outcomes from a given experiment.

Given two events **A** and **B** and given that these events are part of a sample space **S**. This sample space is represented as a set as in the diagram below.



The entire sample space of **S** is given by:

$$S = \{A, B, C\}$$

Remember the following from set theory:

$$C = (A \cup B)'$$
$$A \cup B = A + B - (A \cap B)$$

The different regions of the set **S** can be explained as using the rules of probability.

Rules of Probability

When dealing with more than one event, there are certain rules that we must follow when studying probability of these events. These rules depend greatly on whether the events we are looking at are Independent or dependent on each other.

First acknowledge that

$$P(S) = P(A \cup B \cup C)$$

Multiplication Rule (A∩B)

This region is referred to as 'A intersection B' and in probability; this region refers to the event that both **A** and **B** happen. When we use the word **and** we are referring to multiplication, thus **A and B** can be thought of as **AxB** or (using dot notation which is more popular in probability) **A•B**

If **A** and **B** are dependent events, the probability of this event happening can be calculated as shown below:

$$P(A \cap B) = P(A \cup B) - (P(A\ \text{only}) + P(B\ \text{only}))$$

If **A** and **B** are independent events, the probability of this event happening can be calculated as shown below:

$$P(A \cap B) = P(A) \times P(B)$$

Conditional probability for two independent events can be redefined using the relationship above to become:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{P(A) \times P(B)}{P(A)}$$

$$\mathrm{P(B|A) = P(B)}$$

The above is consistent with the definition of independent events, the occurrence of event **A** in no way influences the occurrence of event **B**, and so the probability that event **B** occurs given that event **A** has occurred is the same as the probability of event **B**.

Additive Rule (AU B)

In probability we refer to the addition operator (+) as **or**. Thus when we want to we want to define some event such that the event can be A or B, to find the probability of that event:

$$\mathrm{P(A+B) = P(A \cup B)}$$

$$\mathrm{P(A \cup B) = A + B - P(A \cap B)}$$

Thus it follows that:

$$\mathrm{P(A+B) = P(A) + P(B) - P(A \cap B)}$$

But remember from set theory that and from the way we defined our sample space above:

$$\mathrm{(A \cup B) = 1 - (A \cup B)'}$$

and that:

$$\mathrm{(A \cup B)' = C}$$

So we can now redefine out event as

$$\mathrm{P(A+B) = 1 - P(A \cup B)'}$$

$$\mathrm{P(A+B) = 1 - P(C)}$$

The above is sometimes referred to as the subtraction rule.

Mutual Exclusivity

Certain special pairs of events have a unique relationship referred to as mutual exclusivity. Two events are said to be mutually exclusive if they can't occur at the same time. For a given sample space, its either one or the other but not both. As a consequence, mutually exclusive events have their probability defined as follows:

$$\mathrm{P(A) + P(B) = 1}$$

An example of mutually exclusive events are the outcomes of a fair coin flip. When you flip a fair coin, you either get a head or a tail but not both, we can prove that these events are mutually exclusive by adding their probabilities:

$$\mathrm{P(head) + P(tail)} = \frac{1}{2} + \frac{1}{2} = 1$$

For any given pair of events, if the sum of their probabilities is equal to one, then those two events are mutually exclusive.

Rules of Probability for Mutually Exclusive Events

- Multiplication Rule: From the definition of mutually exclusive events, we should quickly conclude the following:

$$\mathrm{P(A \cap B) = 0}$$

- Addition Rule: As we defined above, the addition rule applies to mutually exclusive events as follows:

$$P(A + B) = 1$$

- Subtraction Rule: From the addition rule above, we can conclude that the subtraction rule for mutually exclusive events takes the form;

$$P(A \cup B)' = 0$$

Conditional Probability for Mutually Exclusive Events

We have defined conditional probability with the following equation:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

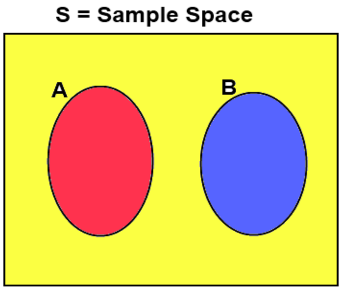
We can redefine the above using the multiplication rule

$$P(A \cap B) = 0$$

hence

$$P(B|A) = \frac{0}{P(A)} = 0$$

Below is a venn diagram of a set containing two mutually exclusive events **A** and **B**.



Probability Distribution

The Poisson random variable satisfies the following conditions:

1. The number of successes in two disjoint time intervals is independent.
2. The probability of a success during a small time interval is proportional to the entire length of the time interval.

Apart from disjoint time intervals, the Poisson random variable also applies to **disjoint regions of space**.

Applications

- the number of deaths by horse kicking in the Prussian army (first application)
- birth defects and genetic mutations
- rare diseases (like Leukemia, but not AIDS because it is infectious and so not independent) - especially in legal cases
- car accidents
- traffic flow and ideal gap distance
- number of typing errors on a page

- hairs found in McDonald's hamburgers
- spread of an endangered animal in Africa
- failure of a machine in one month

The **probability distribution of a Poisson random variable** X representing the number of successes occurring in a given time interval or a specified region of space is given by the formula:

$P(X)=e_{-\mu}\mu_{x}x!$

where

$x=0,1,2,3...$

$e=2.71828$ (but use your calculator's e button)

$\mu=$ mean number of successes in the given time interval or region of space

The **normal** (or **Gaussian**) **distribution** is a very commonly occurring continuous probability distribution—a function that tells the probability of a number in some context falling between any two real numbers. a theoretical frequency distribution represented by a normal curve.

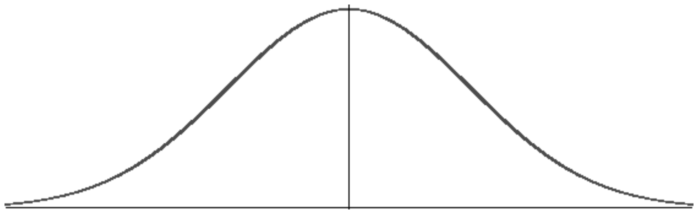


Figure 4.1: A Normal Distribution

The normal distribution is

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

The parameter μ in this formula is the *mean* or *expectation* of the distribution (and also its median and mode). The parameter σ is its standard deviation; its variance is therefore σ^2. A random variable with a Gaussian distribution is said to be **normally distributed** and is called a **normal deviate**.

If $\mu = 0$ and $\sigma = 1$, the distribution is called the **standard normal distribution** or the **unit normal distribution**, and a random variable with that distribution is a **standard normal deviate**.

CHAPTER 6: GRAPHS OF FUNCTIONS

Defining the Graph of a Function

The graph of a function f is the set of all points in the plane of the form (x, f(x)). We could also define the graph of f to be the graph of the equation y = f(x). So, the graph of a function if a special case of the graph of an equation.

Example 1.

Let **$f(x) = x^2 - 3$**.

Recall that when we introduced graphs of equations we noted that if we can solve the equation for y, then it is easy to find points that are on the graph. We simply choose a number for x, then compute the corresponding value of y. Graphs of functions are graphs of equations that have been solved for y!

The graph of f(x) in this example is the graph of $y = x^2 - 3$. It is easy to generate points on the graph. Choose a value for the first coordinate, then evaluate f at that number to find the second coordinate. The following table shows several values for x and the function f evaluated at those numbers.

x	-2	-1	0	1	2
f(x)	1	-2	-3	-2	1

Each column of numbers in the table holds the coordinates of a point on the graph of f.

Exercise 1:

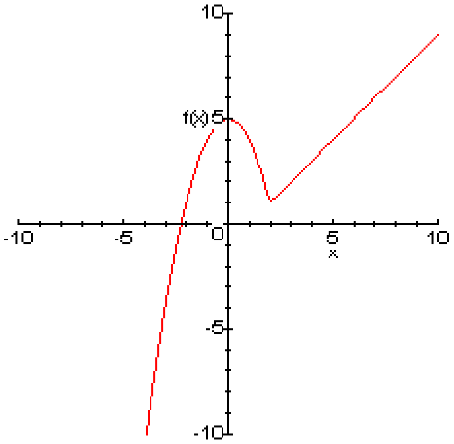
(a) Plot the five points on the graph of f from the table above, and based on these points, sketch the graph of f.

Example 2.

Let f be the piecewise-defined function

$$f(x) = \begin{cases} 5 - x^2, & x \leq 2 \\ x - 1, & x > 2 \end{cases}$$

The graph of f is shown below.



Exercise 2:

Graph the piecewise-defined function

$$f(x) = \begin{cases} 1 - x, & x \le 4 \\ 2x - 11, & x > 4 \end{cases}$$

We have seen that some equations in x and y do *not* describe y as a function of x. The algebraic way see if an equation determines y as a function of x is to solve for y. If there is not a unique solution, then y is not a function of x.

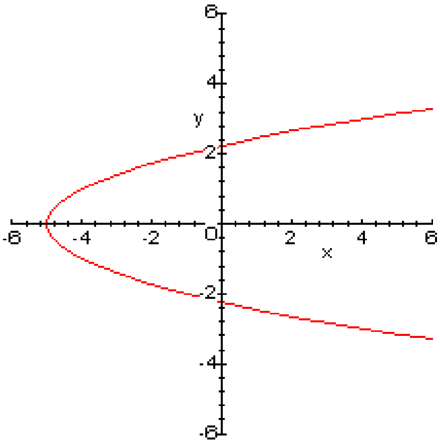
Suppose that we are given the graph of the equation. There is an easy way to see if this equation describes y as a function of x.

Vertical Line Test

A set of points in the plane is the graph of a function if and only if no vertical line intersects the graph in more than one point.

Example 3.

The graph of the equation $\mathbf{y^2 = x + 5}$ is shown below.



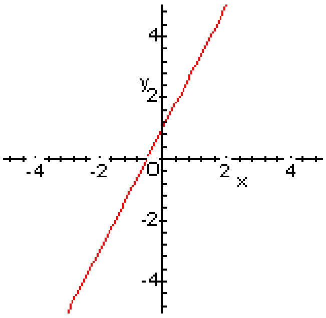
By the vertical line test, this graph is not the graph of a function, because there are many vertical lines that hit it more than once.

Think of the vertical line test this way. The points on the graph of a function f have the form (x, f(x)), so once you know the first coordinate, the second is determined. Therefore, there cannot be two points on the graph of a function with the same first coordinate.

All the points on a vertical line have the same first coordinate, so if a vertical line hits a graph twice, then there are two points on the graph with the same first coordinate. If that happens, the graph is not the graph of a function

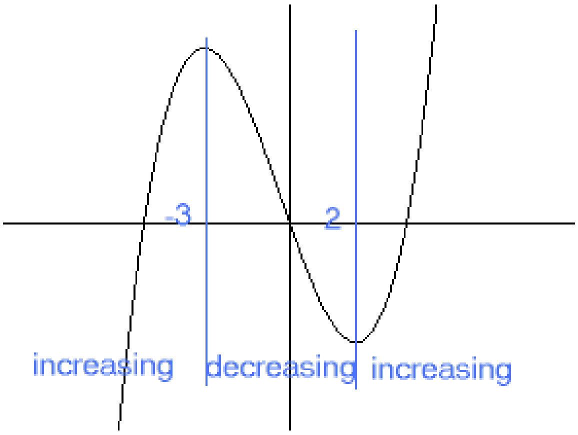
Characteristics of Graphs

Consider the function f(x) = 2 x + 1. We recognize the equation y = 2 x + 1 as the Slope-Intercept form of the equation of a line with slope 2 and y-intercept (0,1).



Think of a point moving on the graph of f. As the point moves toward the right it rises. This is what it means for a function to be *increasing*. Your text has a more precise definition, but this is the basic idea.

The function f above is increasing everywhere. In general, there are intervals where a function is increasing and intervals where it is decreasing.



The function graphed above is decreasing for x between -3 and 2. It is increasing for x less than -3 and for x greater than 2.

Using interval notation, we say that the function is decreasing on the interval (-3, 2) increasing on (-infinity, -3) and (2, infinity)

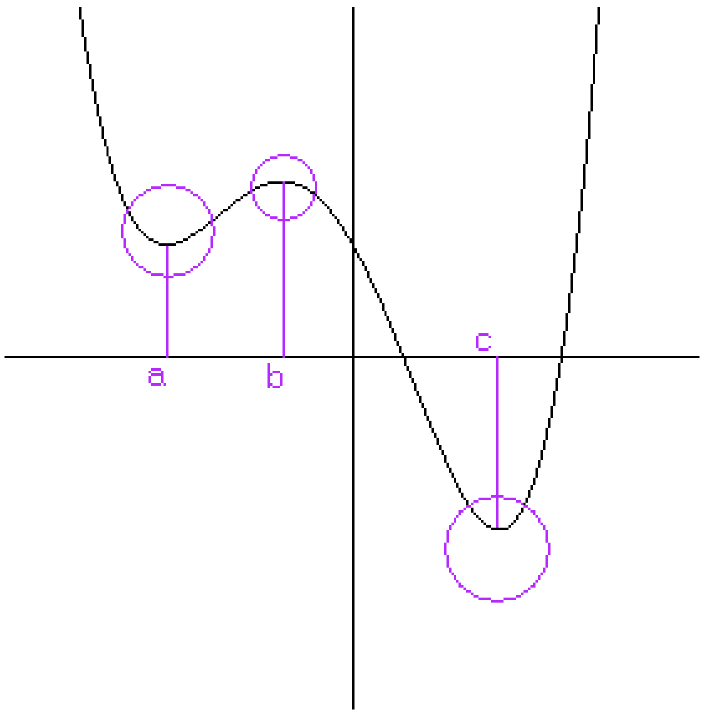
Exercise 3:

Graph the function $f(x) = x^2 - 6x + 7$ and find the intervals where it is increasing and where it is decreasing.

Answer

- decreasing on (-infinity, 3)
- increasing on (3, infinity)

Some of the most characteristics of a function are its **Relative Extreme Values**. Points on the functions graph corresponding to relative extreme values are turning points, or points where the function changes from decreasing to increasing or vice versa. Let f be the function whose graph is drawn below.



f is decreasing on (-infinity, a) and increasing on (a, b), so the point (a, f(a)) is a turning point of the graph. f(a) is called a **relative minimum** of f. Note that f(a) is *not* the smallest function value, f(c) is. However, if we consider only the portion of the graph in the circle above a, then f(a) is the smallest second coordinate. Look at the circle on the graph above b. While f(b) is not the largest function value (this function does not have a largest value), if we look only at the portion of the graph in the circle, then the point (b, f(b)) is above all the other points. So, f(b) is a **relative maximum** of f. f(c) is another relative minimum of f. Indeed, f(c) is the **absolute minimum** of f, but it is also one of the relative minima.

Here again we are giving definitions that appeal to your geometric intuition. The precise definitions are given in your text.

Approximating Relative Extrema

Suppose a is a number such that f(a) is a relative minimum. In applications, it is often more important to know *where* the function attains its relative minimum than it is to know what the relative minimum is.

For example, $f(x) = x^3 - 4x^2 + 4x$ has a relative minimum of 0. It attains this relative minimum at $x = 2$, so (2,0) is a turning point of the graph of f. We will call the point (2,0) a relative minimum point. In general, a **relative extreme point** is a point on the graph of f whose second coordinate is a relative extreme value of f.

Even and Odd Functions

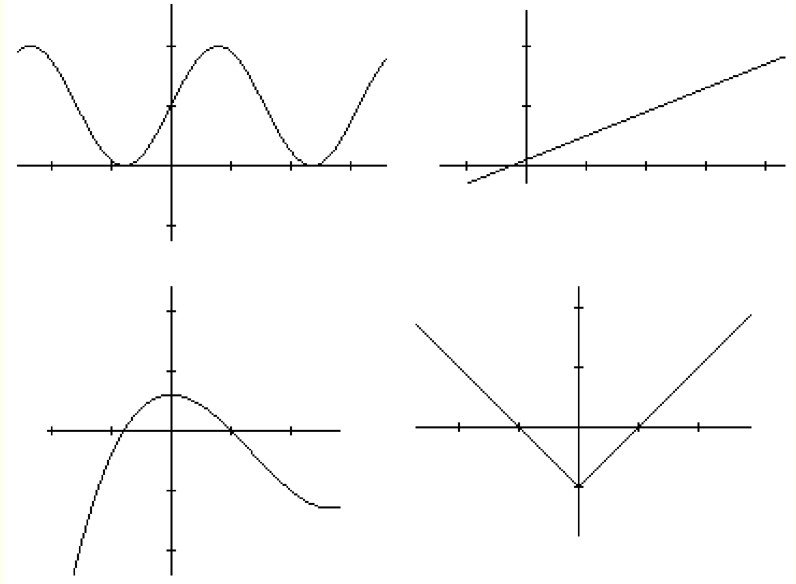
A function f is **even** if its graph is symmetric with respect to the y-axis. This criterion can be stated algebraically as follows: f is even if f(-x) = f(x) for all x in the domain of f. For example, if you evaluate f at 3 and at -3, then you will get the same value if f is even.

A function f is **odd** if its graph is symmetric with respect to the origin. This criterion can be stated algebraically as follows: f is odd if f(-x) = -f(x) for all x in the domain of f. For example, if you evaluate f at 3, you get the negative of f(-3) when f is odd.

Continuous Functions

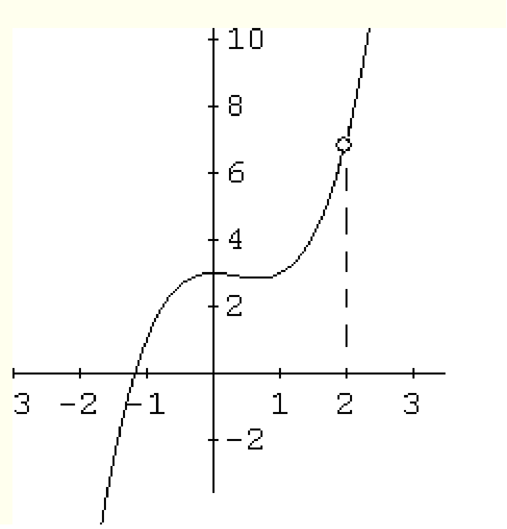
Introduction and Definition of Continuous Functions

We first start with graphs of several continuous functions. The functions whose graphs are shown below are said to be continuous since these graphs have no "breaks", "gaps" or "holes".

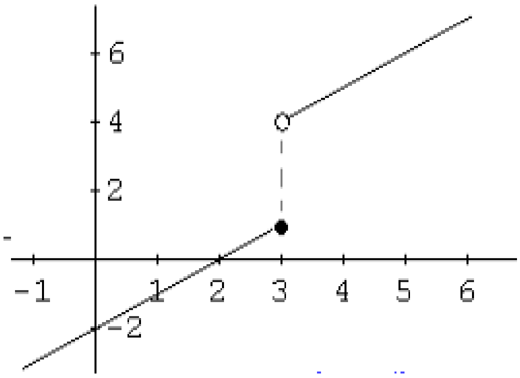


We now present examples of dicontinuous functions. These graphs have: breaks, gaps or points at which they are undefined.

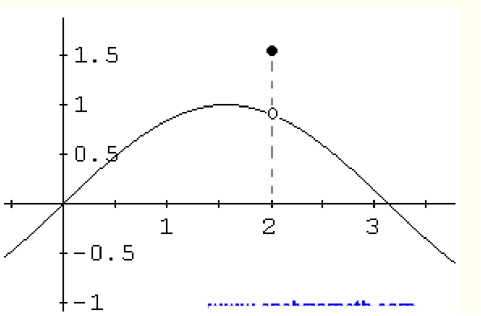
In the graphs below, the function is undefined at x = 2. The graph has a hole at x = 2 and the function is said to be discontinuous.



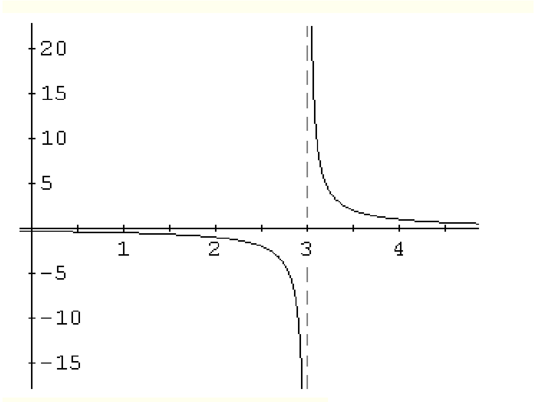
In the graphs below, the limits of the function to the left and to the right are not equal and therefore the limit at x = 3 does not exist. The function is said to be discontinuous.



The limits of the function at x = 2 exists but it is not equal to the value of the function at x = 2. This function is also discontinuous.



The limits of the function at x = 3 does does not exist since to the left and to the right of 3 the function either increases or decreases undefinitely. This function is also discontinuous.



Taking into consideration all the information gathered from the examples of continuous and discontinuous functions shown above, we define a continuous functions as follows:

Function f is continuous at a point **a** if the following conditions are satisfied.

1. f(a) is defined
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

Interpolation and Extrapolation

Definitions

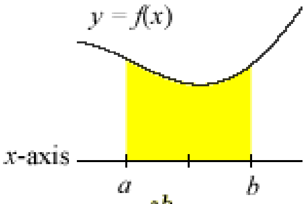
- **Interpolation** is the process of obtaining a value from a graph or table that is located between major points given, or between data points plotted. A ratio process is usually used to obtain the value. *Interpolation allows you to add new data points between pairs of existing data points*
- **Extrapolation** is the process of obtaining a value from a chart or graph that extends beyond the given data. The "trend" of the data is extended past the last point given and an estimate made of the value. *Extrapolation allows you to add data points that extend beyond the beginning or ending values of your data range.*

Area under a Curve

The area between the graph of $y = f(x)$ and the x-axis is given by the definite integral below. This formula gives a positive result for a graph above the x-axis, and a negative result for a graph below the x-axis.

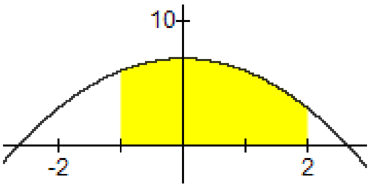
Note: If the graph of $y = f(x)$ is partly above and partly below the x-axis, the formula given below generates the net area. That is, the area above the axis minus the area below the axis.

Formula:



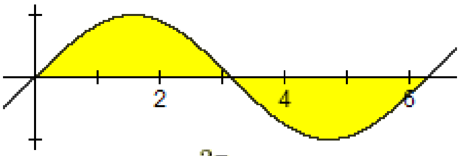
$$\text{Area} = \int_a^b f(x)dx$$

Example 1: Find the area between $y = 7 - x^2$ and the x-axis between the values $x = -1$ and $x = 2$.



$$\begin{aligned}\text{Area} &= \int_{-1}^{2} \left(7 - x^2\right) dx \\ &= \left(7x - \frac{1}{3}x^3\right)\Bigg|_{-1}^{2} \\ &= \left[7 \cdot 2 - \frac{1}{3}(8)\right] - \left[7(-1) - \frac{1}{3}(-1)\right] \\ &= 18\end{aligned}$$

Example 2: Find the net area between $y = \sin x$ and the x-axis between the values $x = 0$ and $x = 2\pi$.



$$\begin{aligned}\text{Net area} &= \int_0^{2\pi} \sin x dx \\ &= (-\cos x)|_0^{2\pi} \\ &= (-1) - (-1) \\ &= 0\end{aligned}$$

Area between Curves

The area between curves is given by the formulas below.

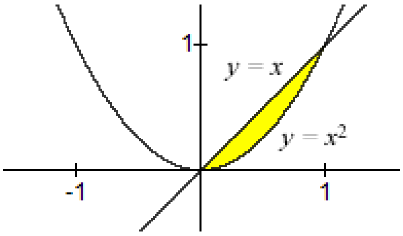
Formula 1: $$\text{Area} = \int_a^b |f(x) - g(x)|\, dx$$

for a region bounded above and below by $y = f(x)$ and $y = g(x)$, and on the left and right by $x = a$ and $x = b$.

Formula 2: $$\text{Area} = \int_c^d |f(y) - g(y)|\, dy$$

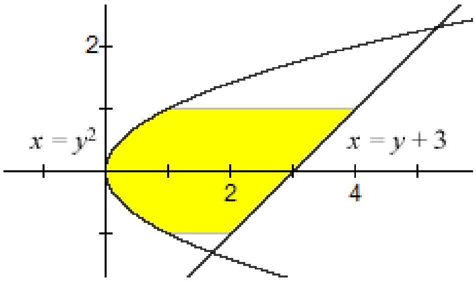
for a region bounded left and right by $x = f(y)$ and $x = g(y)$, and above and below by $y = c$ and $y = d$.

Example 1: Find the area between $y = x$ and $y = x^2$ from $x = 1$ to $x = 2$.



$$\begin{aligned}\text{Area} &= \int_0^1 \left|x - x^2\right| dx \\ &= \int_0^1 \left(x - x^2\right) dx \\ &= \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^1 \\ &= \frac{1}{6}\end{aligned}$$

Example 2: Find the area between $x = y + 3$ and $x = y^2$ from $y = -1$ to $y = 1$.



$$\begin{aligned}
\text{Area} &= \int_{-1}^{1} \left| y + 3 - y^2 \right| dy \\
&= \int_{-1}^{1} \left(y + 3 - y^2 \right) dy \\
&= \left(\frac{1}{2} y^2 + 3y - \frac{1}{3} y^3 \right) \Big|_{-1}^{1} \\
&= \left(\frac{1}{2} + 3 - \frac{1}{3} \right) - \left(\frac{1}{2} - 3 + \frac{1}{3} \right) \\
&= \frac{16}{3}
\end{aligned}$$

Inequality

In mathematics, an **inequality** is a relation that holds between two values when they are different (see also: equality).

- The notation $a \neq b$ means that a is **not equal to** b.

It does not say that one is greater than the other, or even that they can be compared in size.

If the values in question are elements of an ordered set, such as the integers or the real numbers, they can be compared in size.

- The notation $a < b$ means that a is **less than** b.
- The notation $a > b$ means that a is **greater than** b.

In either case, a is not equal to b. These relations are known as **strict inequalities**. The notation $a < b$ may also be read as "a is strictly less than b".

In contrast to strict inequalities, there are two types of inequality relations that are not strict:

- The notation $a \leq b$ means that a is **less than or equal to** b (or, equivalently, **not greater than** b, or **at most** b).
- The notation $a \geq b$ means that a is **greater than or equal to** b (or, equivalently, **not less than** b, or **at least** b)

An additional use of the notation is to show that one quantity is much greater than another, normally by several orders of magnitude.

- The notation $a \ll b$ means that a is **much less than** b. (In measure theory, however, this notation is used for absolute continuity, an unrelated concept.)
- The notation $a \gg b$ means that a is **much greater than** b.

Solving Inequalities

Sometimes we need to solve Inequalities like these:

Symbol	**Words**	**Example**
>	greater than	$x + 3 > 2$
<	less than	$7x < 28$
≥	greater than or equal to	$5 \geq x - 1$
≤	less than or equal to	$2y + 1 \leq 7$

Solving

Our aim is to have x (or whatever the variable is) **on its own** on the left of the inequality sign:

Something like: $x < 5$

or: $y \geq 11$

We call that "solved".

These are things you can do **without affecting** the direction of the inequality:

- Add (or subtract) a number from both sides
- Multiply (or divide) both sides by a **positive** number
- Simplify a side

Example: 3x < 7+3

You can simplify 7+3 without affecting the inequality:

$$3x < 10$$

But these things will change the direction of the inequality ("<" becomes ">" for example):

- Multiply (or divide) both sides by a **negative** number
- Swapping left and right hand sides

Example: 2y+7 < 12

When you swap the left and right hand sides, you must also **change the direction of the inequality**:

12 **>** 2y+7

Common Graph Construction Errors

Confusing independent and dependent variables. The Independent variable is what I change." The size of the Dependent variable Depends on the value of the independent variable.

Putting wrong variables on axes. For a matter of convenience, it is common practice to put the independent variable on the horizontal x-axis (bottom) rather than the vertical y-axis (side) when seeking a relationship to define the dependent variable.

For instance, if one wants to arrive at a relationship describing F as a function of a, then F (the dependent variable) should be plotted on the y-axis. The slope becomes the proportionality constant, F = ma.

Failure to understand the significance of "linearizing" data. When data are non-linear (not in a straight line when graphed), it is best to "linearize" the data. This does not mean to fit the curved data points with a straight line. Rather, it means to modify one of the variables in some manner such that when the data are graphed using this new data set, the resulting data points will appear to lie in a straight line. For instance, say the data appear to be an inverse function – as x is doubled, y is halved. To linearize the data for such a function plot x versus 1/y. If this is indeed an inverse function, then the plot of x versus 1/y date will be linear.

Failure to properly relate y = mx + b to the linearized data. When plotting, say, distance (on the y-axis) versus time (on the x-axis), the correct relationship between distance and time can be found by relating y to distance, x to time, m to the slope, and b to the y-intercept. For instance, data have been linearized for the function resulting in a straight-line graph when distance is plotted versus time-squared. The slope is 2m/s2 and the y-intercept is 1 m. The correct form of the relationship between all the variables will be distance = (2m/s2)*time2 + 1 m.

Failure to apply appropriate labeling. Each graph should be appropriately labeled; each axis should be similarly labeled with its variable and units (in parentheses). For instance, time (seconds) or distance (meters).

Confusing variables with units. Don't confuse variables (time, resistance, distance) with their units (seconds, ohms, meters).

Connecting data points with straight lines. Never connect even linearized data point-to-point with straight lines. Linearized data sometimes appears not to fit the best-fit line precisely. This is frequently due to errors in the data.

Scaling errors. In physics it is often best to show the origin (0,0) in each graph. While this is not an error, it frequently leads to misinterpretations of relationships. Another problem to avoid is not sticking with a consistent scale on an axis. Values should be equally spaced on graphs. For instance, the distances between 1, 2, 3, and 4 should be the same. To be avoided are situations where the spacing between such numbers is irregular or inconsistent.

Failing to properly interpret horizontal and vertical slopes. When the slope of a line is zero degrees (horizontal), then there is no relationship between the plotted variables. When the slope of a line is 90 degrees (vertical), there is no relationship between the plotted variables. In both these cases, the so-called "dependent" variable does not actually depend upon the independent variable.

Failing to properly interpret a negative slope. A negative slope does NOT imply an inverse relationship. What it does imply is that when x gets larger, y becomes larger but in the negative direction. $y = -mx$ is a negative slope. $xy = m$ or $x = m/y$ is an inverse relationship. Additionally, a negative acceleration does not necessary imply a slowing down of an object. An object moving in the –x direction will, in fact, be increasing in speed.

Failing to properly interpret the y-intercept. The value of the y-intercept represents the value of the dependent variable when the independent variable is zero. Sometimes when lines or curve representing relationships clearly should pass through the origin (0, 0), they do not. The value of the y-intercept is then small but not zero. This is usually the result of data collection errors. Linearized data can be "forced through the origin" by conducting a proportional fit of the data ($y = mx$) rather than a linear fit ($y = mx + b$).

Finding slope using two data points NOT on best-fit line. Finding the slope of the best-fit line from two data points rather than two points on the best-fit line is a common error. This can result in a substantial error in the slope. Use two points on the best-fit line for this process, not data points to find the slope of a line.

Improperly finding the slope using elements of a single coordinate point. When finding the slope of a tangent line, students sometimes will mistakenly take the coordinate point nearest where the tangent line is drawn and from those coordinates attempt to determine the slope. For instance, at (x, y) = (5s, 25m) the data point values are mistakenly used to find a slope of 25m/5s = 5m/s.

Failing to recognize trends. Graphs must be properly interpreted through the use of generalizations (e.g., double the mass, and the period increases by 4)

Improperly interpreting the physical meaning of the slope. The units of a slope are a giveaway as far as interpreting the physical meaning of a slope is concerned. The slope is defined as the change in the y value divided by a corresponding change in the x value. Hence, the units are those of the y-axis divided by those of the x-axis. If distance (m) is on the y-axis, and time(s) on the x-axis, then the slope's units will be m/s which is the unit for speed. Similarly, if velocity (m/s) is on the y-axis and time (s) is on the x-axis, then (m/s)/s = m/s2, the unit of acceleration. Slopes can be either positive or negative.

Misinterpreting the area under a best-fit line or curve. To more easily determine what the area under the curve represents, examine the product of the units on the x- and y-axes. For instance, if velocity or speed (m/s) is on the y-axis and time (s) is on the x-axis, the product of the units will be meters that is the unit of displacement or distance. Hence, the area under the curve of a v-t graph is displacement. Similarly, the area under the curve of an a-t graph is velocity.

Not recognizing errors in data. When one graphs data, large errors in measurement usually stand out. They become noticeable when they deviate from the trend of the data. Such data points should not only be questioned, but the data collection process for this datum repeated to see if there is a readily explainable error. Suspect data should be struck through rather than erased in the written record. Update the data file appropriately.

Not understanding the meaning of the best-fit line. A best-fit line is a line which, when drawn, minimizes the sum of the squares of the deviations from the line usually in the y direction. It is the line that most closely approximates the trend of the data.

Statistics is the study of the collection, organization, analysis, interpretation and presentation of data. It deals with all aspects of data, including the planning of data collection in terms of the design of surveys and experiments.

CHAPTER 7: DATA COLLECTION METHODS

Data collection is a process of collecting data using different methodologies. It is a very important topic of statistics as well as mathematics. The data collections are known as, any information collected from some person or some other ways to get news. Data collection can be defined as a term which is used to explain the process of preparing and collecting data, such as, a part of a process improvement. The purpose of data collection is to collect important information to keep on record for further use, to make important decisions about different issues, and to pass vital information on to others.

Introduction to Data Collection Methods

The data collection is to collect important information to keep on record for further use, to make important decisions about different issues, and to pass vital information on to others.
In terms of the method of data collection that will be used for the study, there are mainly two types of data:

- Primary data
- Secondary data

Primary Data
They are collected afresh and for the first time, and thus happens to be original in character. They are the most original data and mostly have not undergone any sort of statistical test.
Secondary Data
They are those that has already been collected by someone else and which has been already been passed through the statistical process. They are not pure and have undergone some treatment at least once.

Data Edition:

After collecting the required data, either from primary or secondary means, the next step leads to edition. Editing is a process by which the data collected is examined to discover any error and mistake before it is presented. It has to be understood beforehand itself to what degree the accuracy is needed and to what extent the errors can be tolerated in the inquiry. The editing of secondary data is much simpler than that of primary data.

Types of Data Collection Methods

Data collection simply defines that, how the information was gathered? It is known as data collection. The getting data is may be given us to particular information related to that data.

There are two different methods of data’s are collected in data collection.

- Primary data
- Secondary data.

And these two data collection methods are having some important tools used for collecting data. Such a data collection tools used for the two methods are given below:

Primary data tools are interviews, surveys, direct observations, focusing groups.

Secondary data tools are telephone, mobile phones, e - mails, post cards, etc.

Selecting Appropriate Data Collection Method

As there are many method to collect the data it important that we choose the most appropriate according to the situation provided. So the following factors has to be kept in mind while selecting a particular method:

1. The nature, the scope as well as the object of the enquiry is very important as it will affect the choice of the method.
2. When a method is chosen it's important to check whether there is adequate amount of funds to make it work. If the method is too expensive, it will be very hard to do the experiment.
3. Time is an important factor as decided when the experiment has to end.
4. Precision is also another important factor.

But it must be always remembered theta each method of data collection has its use and none is superior in all situations.

After data collection, the method can be broadly divided into two types

- **Ungrouped data:** They are those data's that are not arranged in any systematic order. It can be arranged only in ascending or descending order. They are also termed as raw data.
- **Grouped data:** They are data's that are presented in the form of frequency distribution.

Classification of Data

The process of arranging data into homogenous group or classes according to some common characteristics present in the data is called classification.
For Example: The process of sorting letters in a post office, the letters are classified according to the cities and further arranged according to streets.

Bases of Classification:
There are four important bases of classification:
Classifications of data
A. According to Nature
1. Quantitative data- information obtained from numeral variables(e.g. age, bills, etc)

2.Qualitative Data- information obtained from variables in the form of categories, characteristics names or labels or alphanumeric variables (e.g. birthdays, gender etc.)

B. According to Source

1. Primary data- first- hand information (e.g. autobiography, financial statement)
2. Secondary data- second-hand information (e.g. biography, weather forecast from news papers)

C. According to Measurement

1. Discrete data- countable numerical observation.
   - Whole numbers only
   - has an equal whole number interval
   - obtained through counting(e.g. corporate stocks, etc.)
2. Continuous data-measurable observations.
   - decimals or fractions
   - obtained through measuring(e.g. bank deposits, volume of liquid etc.)

D. According to Arrangement

1. Ungrouped data- raw data
   - no specific arrangement
2. Grouped Data - organized set of data
   - at least 2 groups involved
   - arranged

Data Tabulation and Presentation

The process of placing classified data into tabular form is known as tabulation. A table is a symmetric arrangement of statistical data in rows and columns. Rows are horizontal arrangements whereas columns are vertical arrangements. It may be simple, double or complex depending upon the type of classification.

one of the most important aspect in any statistical investigation is the manner by which the researcher presents the data

various modes of data presentation are:

a) Textual-data are presented in the form of texts, phrases or paragraphs. Common among newspapers and reports
b) tabular – a more reliable and effective way of showing relationships or comparisons of data
c) graphical-the most effective way of presenting data through the use of statistical graph

Tables are used to present numerical data in a wide variety of publications from newspapers, journals and textbooks to the sides of grocery packets. They are the format in which most numerical data are initially stored and analysed and are likely to be the means you use to organise data collected during experiments and dissertation research. However, when writing up your work you will have to make a decision about whether a table is the best way of presenting the data, or if it would be easier to understand if you were to use a graph or chart.

This section of the guide identifies the appropriate uses of tables, and discusses some design issues for constructing clear tables which are easy to interpret. The points covered in this guide apply equally to primary data that you have collected yourself, and to data that you have found in secondary sources and which you wish to include in your work. The latter may already be presented as a table in the original work but you do not have to reproduce it exactly. It may be that you only require an extract from the table to support your argument, or that the design of the table could be improved, or that you wish to merge information from two different publications. There is no problem in doing any of these as long as you ensure that you reference the original source of the data in your table.

When to use tables

Tables are an effective way of presenting data:

- when you wish to show how a single category of information varies when measured at different points (in time or space). For example, a table would be an appropriate way of showing how the category unemployment rate varies between different countries in the EU (different points in space);
- When the dataset contains relatively few numbers. This is because it is very hard for a reader to assimilate and interpret many numbers in a table . In particular, avoid the use of complex tables in talks and presentations when the audience will have a relatively short time to take in the information and little or no opportunity to review it at a later stage;
- When the precise value is crucial to your argument and a graph would not convey the same level of precision. For example, when it is important that the reader knows that the result was 2.48 and not 2.45;
- When you don’t wish the presence of one or two very high or low numbers to detract from the message contained in the rest of the dataset. For example if you are presenting information about the annual profits of an organisation and don’t want the underlying variability from one year to the next to be swamped by a large loss in a particular year.

Table design

In order to ensure that your table is clear and easy to interpret there are a number of design issues that need to be considered. These are listed below:

- Since tables consist of rows and columns of information it is important to consider how the data are arranged between the two. Most people find it easier to identify patterns in numerical data by reading down a column rather than across a row. This means that you should plan your row and column categories to ensure that the patterns you wish to highlight are revealed in the columns. It is also easier to interpret the data if they arranged according to their magnitude so there is numerical progression down the columns, although this may not always be possible.
- If there are several columns or categories of information a table can appear complex and

become hard to read. It also becomes more difficult to list the data by magnitude since the order that applies to one column may not be the same for others. In such cases you need to decide which column contains the most important trend and this should be used to structure the table. If the columns are equally important it is often better to include two or more simple tables rather than using a single more complex one.

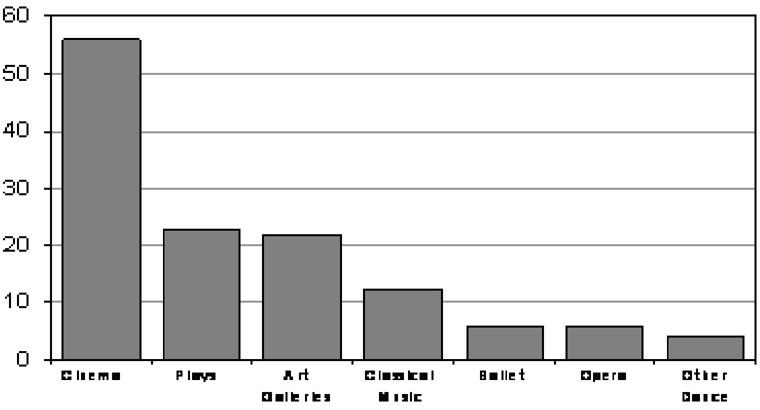
- Numbers in tables should be presented in their most simple format. This may mean rounding up values to avoid the use of decimal places, stating the units (e.g. £4.6 million rather than £4,600,000) or using scientific notation (e.g. 6.315 x 10-2 rather than 0.06315).
- All tables should be presented with a title that contains enough detail that a reader can understand the content without needing to consult the accompanying text. There should also be information about the source of the data being used; this may be a reference to a book or journal, or could indicate that the data are results from an experiment carried out on a particular date.
- Where more than one table is being presented it is standard practice to give each one a unique reference number, and in larger pieces of work, such as dissertations, a list of tables with their page number is usually provided in addition to the contents page.
- The formatting of the table should not resemble a spreadsheet where each entry is bounded by a box since this makes it difficult to read across rows or down columns. However, the design of the table should help the reader interpret the data and so the use of lines and/or bold text to separate headings from the body of data, or highlighting/shading specific rows or may be effective. Avoid large gaps between columns since this also makes it difficult to read along a row.

Graphs are a good means of describing, exploring or summarising numerical data because the use of a visual image can simplify complex information and help to highlight patterns and trends in the data. They are a particularly effective way of presenting a large amount of data but can also be used instead of a table to present smaller datasets. There are many different graph types to choose from and a critical issue is to ensure that the graph type selected is the most appropriate for the data. Having done this, it is then essential to ensure that the design and presentation of the graph help the reader or audience interpret the data.

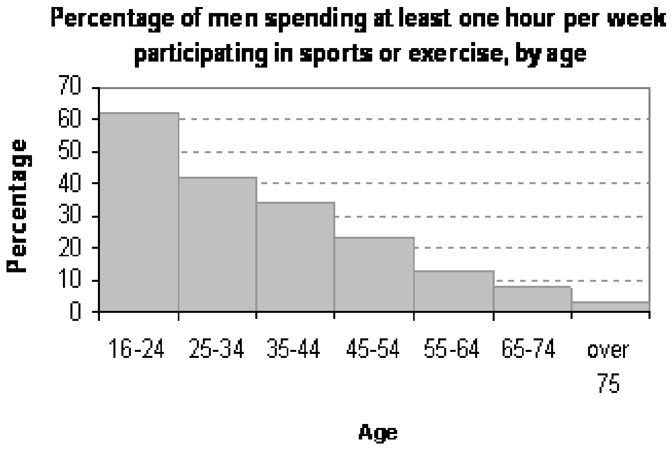
A summary of the types of data that can be presented in the most common types of graphs is provided below and this is followed by some general guidelines for designing readily understandable graphs. There is more detailed information on the uses and good design of particular types of graph in the companion study guides covering bar charts, histograms, pie charts, line graphs and scatter plots available from the Student Learning Centre.

Types of Graph

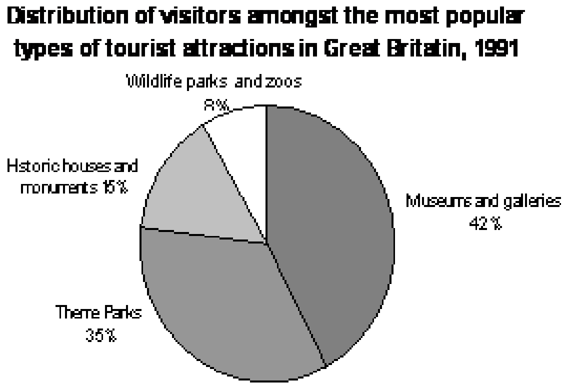
Bar charts are one of the most commonly used types of graph and are used to display and compare the number, frequency or other measure (e.g. mean) for different discrete categories or groups. The graph is constructed such that the heights or lengths of the different bars are proportional to the size of the category they represent. Since the x-axis (the horizontal axis) represents the different categories it has no scale. The y-axis (the vertical axis) does have a scale and this indicates the units of measurement. The bars can be drawn either vertically or horizontally depending upon the number of categories and length or complexity of the category labels. There are various ways in which bar charts can be constructed and this makes them a very flexible chart type. For example, if there is more than one set of values for each category then grouped or component bar charts can be used to display the data. Further details about each of these different types of bar chart can be found in the associated study guide Bar Charts.



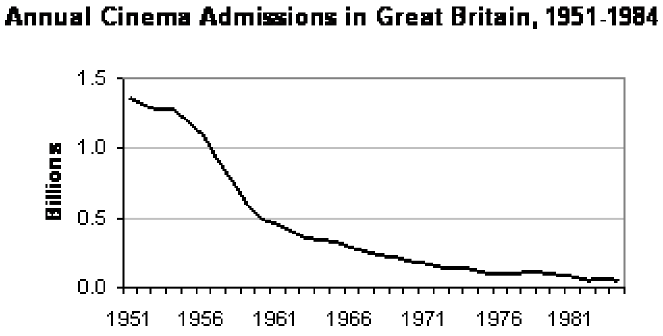
Histograms are a special form of bar chart where the data represent continuous rather than discrete categories. For example a histogram could be used to present details of the average number of hours exercise carried out by people of different ages because age is a continuous rather than a discrete category. However, because a continuous category may have a large number of possible values the data are often grouped to reduce the number of data points. For example, instead of drawing a bar for each individual age between 0 and 65, the data could be grouped into a series of continuous age ranges such as 16-24, 25-34, 35-44 etc. Unlike a bar chart, in a histogram both the x- and y-axes have a scale. This means that it is the area of the bar that is proportional to the size of the category represented and not just its height Further information on constructing histograms is available in the associated study guide Histograms.



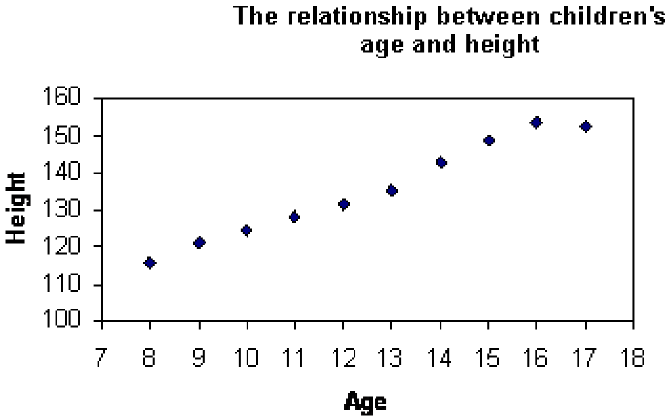
Pie charts are a visual way of displaying how the total data are distributed between different categories. The example here shows the proportional distribution of visitors between different types of tourist attractions. Similar uses of a pie chart would be to show the percentage of the total votes received by each party in an election. Pie charts should only be used for displaying nominal data (i.e. data that are classed into different categories). They are generally best for showing information grouped into a small number of categories and are a graphical way of displaying data that might otherwise be presented as a simple table. The study guide Pie Charts gives more details about designing pie charts and using them to compare data.


Distribution of visitors amongst the most popular types of tourist attractions in Great Britain, 1991

Line graphs are usually used to show time series data – that is how one or more variables vary over a continuous period of time. Typical examples of the types of data that can be presented using line graphs are monthly rainfall and annual unemployment rates. Line graphs are particularly useful for identifying patterns and trends in the data such as seasonal effects, large changes and turning points. As well as time series data, line graphs can also be appropriate for displaying data that are measured over other continuous variables such as distance. For example, a line graph could be used to show how pollution levels vary with increasing distance from a source, or how the level of a chemical varies with depth of soil. However, it is important to consider whether the data have been collected at sufficiently regular intervals so that estimates made for a point lying half-way along the line between two successive measurements would be reasonable. In a line graph the x-axis represents the continuous variable (for example year or distance from the initial measurement) whilst the y-axis has a scale and indicates the measurement. Several data series can be plotted on the same line chart and this is particularly useful for analysing and comparing the trends in different datasets.


Annual Cinema Admissions in Great Britain, 1951-1984

Scatter plots are used to show the relationship between pairs of quantitative measurements made for the same object or individual. For example, a scatter plot could be used to present information about the examination and coursework marks for each of the students in a class. In the example here, the paired measurements are the age and height of children in 1837. In a scatter plot a dot represents each individual or object (child in this case) and is located with reference to the x-axis and y-axis, each of which represent one of the two measurements. By analysing the pattern of dots that make up a scatter plot it is possible to identify whether there is any systematic or causal relationship between the two measurements. For example, in this case it is clear from the upward trending pattern of dots that children's height increases with age. Regression lines can also be added to the graph and used to decide whether the relationship between the two sets of measurements can be explained or if it is due to chance.

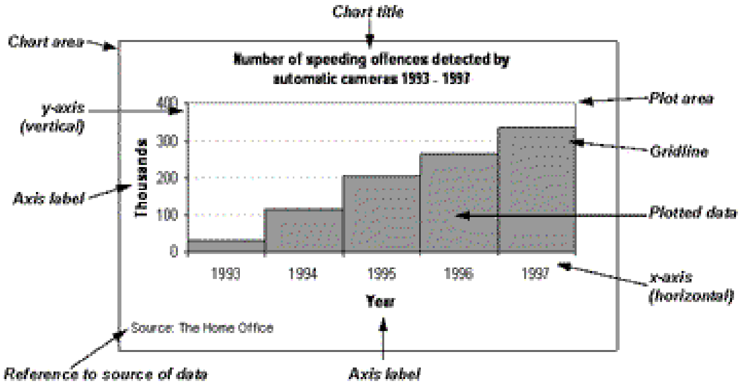


Good graph design

Although there are many different types of graph, there are a number of elements that are common to the majority of them such as axes. This section provides some general guidelines to help you design your graph and ensure that you apply these elements in a way that will help the reader or audience interpret the data you are presenting.

Components of a graph

The different components of a graph are identified in the diagram on the next page and this is followed by a description that highlights some of the specific design and presentation issues related to each component.

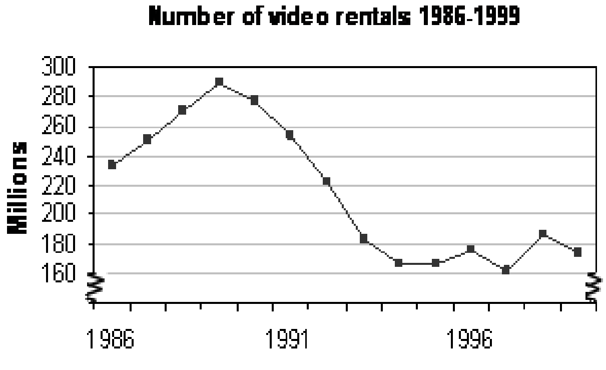


The chart area defines the boundary of all the elements related to the graph including the plot itself and any headings and explanatory text. It emphasises that these elements need to be considered together and that they are separate from the surrounding text. The boundary of the chart area can be imaginary rather than defined by a frame.

The plot area is the region containing the data. It is bounded by the x- and y-axes to the bottom and left side. The frame can be completed by drawing around the top and right sides too, but this is not essential.

The x-axis is the horizontal line that defines the base of the plot area. Depending upon which type of graph is being considered different locations on the x -axis represent either different categories (such as years) or different positions along a numerical scale (such as temperature or income). Details are placed just below the x-axis and an axis label is usually provided to clarify the units of measurement. However, if the category details are mentioned somewhere else such as in the title of the graph, or are very obvious (such as years) then it is not necessary to include an axis label.

The y-axis is the vertical line that usually defines the left side of the plot area, but if more than one variable is being plotted on the graph then the vertical lines on both the left and right sides of the plot area may be used as y-axes. The y-axis always has a numerical scale and is used to show values such as counts, frequencies or percentages. Intervals on the scale are marked by numbers and tick marks, indicating the major divisions, to the left of the y-axis. Like the x-axis, the y-axis usually has a label that provides details of the units of measurement. The label is often written vertically to follow the line of the y-axis but can instead be placed just above the top of the y-axis.
In order to best highlight a trend in the data, it may be necessary to start the y-axis scale at a point other than zero. In such cases the starting value on the y-axis should be clearly labelled and the readers' attention drawn to the non-zero start by breaking the y-axis just below the first value as shown in the example opposite.



Gridlines are the vertical and horizontal lines placed within the plot area to help read values from the graph. The gridlines should be subtle and not detract from the data. In the case of simple graphs it is not always necessary to include them. Gridlines are usually drawn at regular intervals based on the major divisions of the y-axis scale.

Title - All graphs should include a title that summarises what the graph shows. The title should identify what is being described (e.g. speeding offences detected by automatic cameras) and the units of measurements (e.g. percentages, total number, frequency). The title may be placed within the chart area, as in the example above, or above or below the chart.

CHAPTER 8: MEASURES OF CENTRAL TENDENCY

Introduction to central tendency

A **central tendency** (or, more commonly, a **measure of central tendency**) is a central value or a typical value for a probability distribution. It is occasionally called an average or just the **center** of the distribution. The most common measures of central tendency are the arithmetic mean, the median and the mode.

The significance property of measures of central tendency

- It represents a single score around which the center of the entire distribution tends to be located.
- It conveys us the shape and nature of the distribution of data.
- It condenses the data to a single value.
- It enhances the comparison between data.

Types and measures of central tendency

The most commonly used measures of central tendency are:

- Mean
- Median
- Mode

Mean:	**Average.** The sum of a set of data divided by the number of data. (Do not round your answer unless directed to do so.)
Median:	The **middle** value, or the mean of the middle two values, when the data is arranged in numerical order. Think of a "median" being in the middle of a highway.
Mode:	The value (number) that appears the **most**. It is possible to have more than one mode, and it is possible to have no mode. If there is no mode-write "no mode", do not write zero (0) .

Calculation in central tendency

Example #1

Find the mean, median and mode for the following data: 5, 15, 10, 15, 5, 10, 10, 20, 25, 15.

(You will need to organize the data.)

5, 5, 10, 10, 10, 15, 15, 15, 20, 25

Mean: $\frac{\text{sum of data}}{\text{number of data}} = \frac{130}{10} = 13$

Median: 5, 5, 10, 10, 10, 15, 15, 15, 20, 25

Listing the data in order is the easiest way to find the median.
The numbers 10 and 15 both fall in the middle.
Average these two numbers to get the median. $\frac{10 + 15}{2} = \mathbf{12.5}$

Mode: Two numbers appear **most often: 10 and 15.**
There are three 10's and three 15's.
In this example there are two answers for the mode.

Dispersion (also called **variability**, **scatter**, or **spread**) denotes how stretched or squeezed a distribution is

Measures of dispersion measure how spread out a set of data is.

Variance and Standard Deviation

The formulae for the variance and standard deviation are given below. μ means the mean of the data.

Variance = $\sigma^2 = \frac{\Sigma\ (x_r - \mu)^2}{n}$

The standard deviation, σ, is the square root of the variance.

What the formula means:
(1) $x_r - \mu$ means take each value in turn and subtract the mean from each value.
(2) $(x_r - \mu)^2$ means square each of the results obtained from step (1). This is to get rid of any minus signs.
(3) $\Sigma(x_r - \mu)^2$ means add up all of the results obtained from step (2).
(4) Divide step (3) by n, which is the number of numbers
(5) For the standard deviation, square root the answer to step (4).

Example

Find the variance and standard deviation of the following numbers: 1, 3, 5, 5, 6, 7, 9, 10 .

The mean = 46/ 8 = 5.75

(Step 1): (1 - 5.75), (3 - 5.75), (5 - 5.75), (5 - 5.75), (6 - 5.75), (7 - 5.75), (9 - 5.75), (10 - 5.75) = -4.75, -2.75, -0.75, -0.75, 0.25, 1.25, 3.25, 4.25

(Step 2): 22.563, 7.563, 0.563, 0.563, 0.063, 1.563, 10.563, 18.063

(Step 3): 22.563 + 7.563 + 0.563 + 0.563 + 0.063 + 1.563 + 10.563 + 18.063 = 61.504

(Step 4): n = 8, therefore variance = 61.504/ 8 = 7.69 (3sf)

(Step 5): standard deviation = 2.77 (3sf)

Grouped Data

There are many ways of writing the formula for the standard deviation. The one above is for a basic list of numbers. The formula for the variance when the data is grouped is as follows. The standard deviation can be found by taking the square root of this value.

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f}$$

$$\text{Variance, } \sigma^2 = \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2$$

Example

The table shows marks (out of 10) obtained by 20 people in a test

Mark (x)	Frequency (f)
1	0
2	1
3	1
4	3
5	2
6	5
7	5
8	2
9	0
10	1

Work out the variance of this data.

In such questions, it is often easiest to set your working out in a table:

fx	fx^2
0	0
2	4
3	9
12	48
10	50
30	180
35	245
16	128
0	0
10	100

$\Sigma f = 20$
$\Sigma fx = 118$
$Sfx^2 = 764$

$$\text{variance} = \frac{\Sigma fx^2}{\Sigma f} - \frac{(\Sigma fx)^2}{(\Sigma f)^2}$$

$$= \frac{764}{20} - \frac{(118)^2}{(20)^2}$$

$$= 38.2 - 34.81 = \underline{3.39}$$

CHAPTER 9: MEASURES OF DISPERSION

Introduction to measure of Dispersion

The **Absolute Measure** of dispersion is basically the measure of variation from the mean such as standard deviation. On the other hand the **relative measure** of dispersion is basically the position of a certain variable with reference to or as compared with the other variables. Such as the percentiles or the z-score.

Absolute measures of Dispersion are expressed in same units in which original data is presented but these measures cannot be used to compare the variations between the two series. Relative measures are not expressed in units but it is a pure number. It is the ratios of absolute dispersion to an appropriate average such as co-efficient of Standard Deviation or Co-efficient of Mean Deviation.

What is Dispersion? Simplest meaning that can be attached to the word 'dispersion' is a lack of uniformity in the sizes or quantities of the items of a group or series. According to Reiglemen, "Dispersion is the extent to which the magnitudes or quantities of the items differ, the degree of diversity." The word dispersion may also be used to indicate the spread of the data.
In all these definitions, we can find the basic property of dispersion as a value that indicates the extent to which all other values are dispersed about the central value in a particular distribution.

Properties of a good measure of Dispersion

There are certain pre-requisites for a good measure of dispersion:

1. It should be simple to understand.
2. It should be easy to compute.
3. It should be rigidly defined.
4. It should be based on each individual item of the distribution.
5. It should be capable of further algebraic treatment.
6. It should have sampling stability.
7. It should not be unduly affected by the extreme items.

Types of Dispersion

The measures of dispersion can be either 'absolute' or "relative".
Absolute measures of dispersion are expressed in the same units in which the original data are expressed. For example, if the series is expressed as Marks of the students in a particular subject; the absolute dispersion will provide the value in Marks. The only difficulty is that if two or more series are expressed in different units, the series cannot be compared on the basis of dispersion.
'Relative' or 'Coefficient' of dispersion is the ratio or the percentage of a measure of absolute dispersion to an appropriate average. The basic advantage of this measure is that two or more series can be compared with each other despite the fact they are expressed in different units.
Theoretically, 'Absolute measure' of dispersion is better. But from a practical point of view,

relative or coefficient of dispersion is considered better as it is used to make comparison between series.

Absolute Measures

- Range
- quartile Deviation
- Mean Deviation
- Standard Deviation
- Lorenz Curve

Relative Measure

- Co-efficient of Range
- Co-efficient of Quartile Deviation
- Co-efficient of mean Deviation
- Co-efficient of Variation.

Methods of Dispersion and Calculations

Methods of studying dispersion are divided into two types :

(i) Mathematical Methods: We can study the ‘degree’ and ‘extent’ of variation by these methods. In this category, commonly used measures of dispersion are :

(a) Range
(b) Quartile Deviation
(c) Average Deviation
(d) Standard deviation and coefficient of variation.

(ii) Graphic Methods: Where we want to study only the extent of variation, whether it is higher or lesser a Lorenz-curve is used.

Quartiles, deciles, percentiles and S.I.R.

The mean and median both describe the 'center' of a distribution. This is usually what you want to summarize about a set of marks, but occasionally a different part of the distribution is of more interest.

For example, you might want to describe a typical mark for a 'good' or 'weak' student.

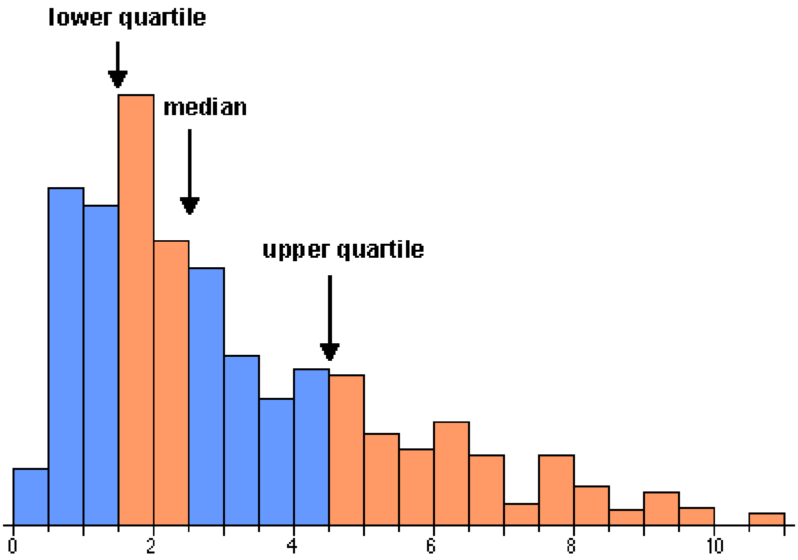
Quartiles

The median of a distribution splits the data into two equally-sized groups. In the same way, the **quartiles** are the three values that split a data set into **four** equal parts. Note that the 'middle' quartile is the median.

The upper quartile describes a 'typical' mark for the top half of a class and the lower quartile is a 'typical' mark for the bottom half of the class.

The quartiles are closely related to the histogram of a data set. Since area equals the proportion of values in a histogram, the quartiles split the histogram into four approximately equal areas.

(The relationship is only approximate if the quartiles do not coincide with histogram bin boundaries.)



Deciles
In a similar way, the **deciles** of a distribution are the **nine** values that split the data set into **ten** equal parts.

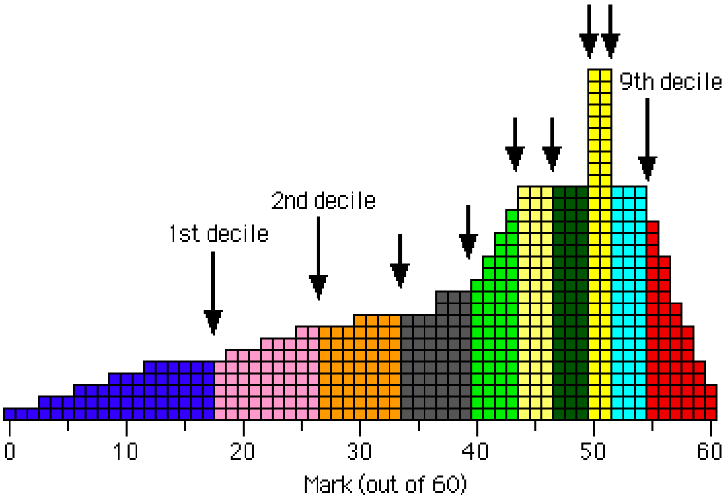
You should not try to calculate deciles from small data sets -- a single class of marks is too small to get useful values since the extreme deciles are very variable. However the deciles can be useful descriptions for larger data sets such as national distributions for marks from standard tests.

Deciles for the distribution and for individual students
The term 'decile' is used in two different contexts. It is confusing that the same word is used in both ways, so be careful!

When applied to a distribution (a large group of marks), there are **nine** deciles, each of which is a **mark**.

A student whose mark is below the first decile is said to be **in decile 1**. Similarly, a student whose marks is between the first and second deciles is **in decile 2**, ... and a student whose marks is above the ninth decile is **in decile 10**. When applied to individual students, the term 'decile' is therefore a **number between 1 and 10**.

For example, the histogram below shows the distribution of marks in a test (out of 60) that was attempted by 600 students. Each student's mark is represented by a square in the histogram.



The nine deciles split the students into 10 groups of 60.

The first decile is 17.5 so the weakest tenth of the students in the class had a mark below this. This decile therefore summarises the performance of the weakest students.

Students with marks below 17.5 are said to be in decile 1. Those with marks between 17.5 and 26.5 are in decile 2, and so on, up to students with marks higher than 54.5 who are in decile 10.

Details

Unfortunately there is no commonly accepted precise definition for the lower and upper quartiles -- different software (and indeed different statisticians!) use slightly different values. One simple definition is that the lower quartile is the median of the lower half of the data (excluding the middle value if there is an even number of values) with a similar definition for the upper quartile.

In practice, the precise definition is of little practical importance, especially for large data sets. The main thing to remember is to be **consistent** with your definition if you are comparing several data sets.

There are similar problems with precisely defining deciles but again the precise definition used should not affect your interpretation of the data.

In practice, you are advised to use the functions built into Excel to evaluate quartiles and deciles.

Percentiles

In a similar way, the **percentiles** of a distribution are the **99** values that split the data set into **a hundred** equal parts. These percentiles can be used to categorise the individuals into percentile 1, ..., percentile 100.

A very large data set is required before the extreme percentiles can be estimated with any accuracy. (The 'random' variability in marks is especially noticeable in the extremes of a data set.)

Quartiles, etc. in Excel

Excel has a built-in function to evaluate the quartiles of a column of marks. If the marks are contained in the cells A1 to A25 of a spreadsheet, the formula "=QUARTILE(A1:A25, 1)" will calculate the lower quartile of the distribution of marks. If the second parameter to the function is 2 or 3, the median or upper quartile will be shown.

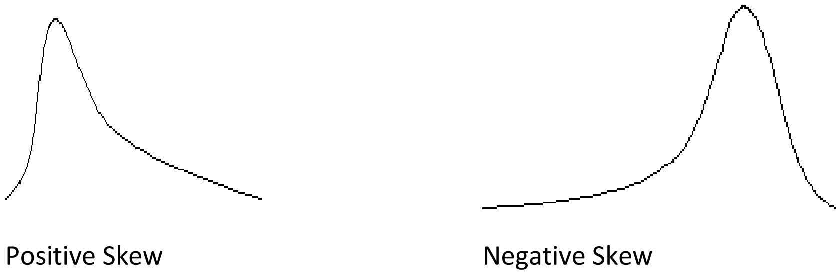
In a similar way, the function "=PERCENTILE (A1:A25, 5)" will evaluate the 5th percentile of the distribution, etc.

Skewness and kurtosis in Dispersion interplatation

Skewness is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point. **Kurtosis** is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution.

Skew, or skewness, can be mathematically defined as the averaged cubed deviation from the mean divided by the standard deviation cubed. If the result of the computation is greater than zero, the distribution is positively skewed. If it's less than zero, it's negatively skewed and equal to zero means it's symmetric. Negatively skewed distributions have what statisticians call a long left tail, which for investors can mean a greater chance of extremely negative outcomes. Positive skew would mean frequent small negative outcomes, and extremely bad scenarios are not as likely.

A nonsymmetrical or skewed distribution occurs when one side of the distribution does not mirror the other. Applied to investment returns, nonsymmetrical distributions are generally described as being either positively skewed (meaning frequent small losses and a few extreme gains) or negatively skewed (meaning frequent small gains and a few extreme losses).



Positive Skew Negative Skew

Figure 2.4

For positively skewed distributions, the mode (point at the top of the curve) is less than the median (the point where 50% are above/50% below), which is less than the arithmetic mean (sum of observations/number of observations). The opposite rules apply to negatively skewed distribution: mode is greater than median, which is greater than arithmetic mean.

Positive: Mean > Median > Mode   Negative: Mean < Median < Mode

Notice that by alphabetical listing, it's mean à median à mode. For positive skew, they are separated with a greater than sign, for negative, less than.

Kurtosis refers to the degree of peak in a distribution. More peak than normal (leptokurtic) means that a distribution also has fatter tails and that there are more chances of extreme outcomes compared to a normal distribution.

The kurtosis formula measures the degree of peak. Kurtosis equals three for a normal distribution; excess kurtosis calculates and expresses kurtosis above or below 3.

In figure below, the solid line is the normal distribution; the dashed line is leptokurtic distribution.

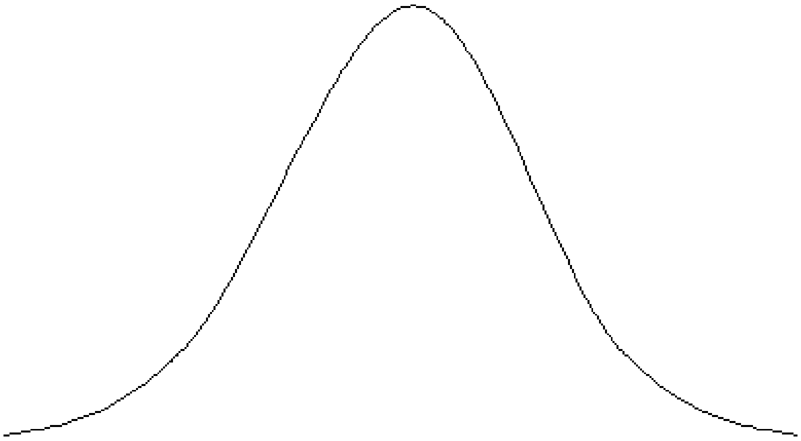


Figure: Kurtosis

Sample Skew and Kurtosis

For a calculated skew number (average cubed deviations divided by the cubed standard deviation), look at the sign to evaluate whether a return is positively skewed (skew > 0), negatively skewed (skew < 0) or symmetric (skew = 0). A kurtosis number (average deviations to the fourth power divided by the standard deviation to the fourth power) is evaluated in relation to the normal distribution, on which kurtosis = 3. Since excess kurtosis = kurtosis - 3, any positive number for excess kurtosis would mean the distribution is leptokurtic (meaning fatter tails and lesser risk of extreme outcomes).

CHAPTER 10: NUMERICAL ANALYSIS

Introduction to Numerical analysis

Numerical analysis is the study of algorithms that use numerical approximation (as opposed to general symbolic manipulations) for the problems of mathematical analysis (as distinguished from discrete mathematics).

Numerical analysis naturally finds applications in all fields of engineering and the physical sciences, but in the 21st century also the life sciences and even the arts have adopted elements of scientific computations. Ordinary differential equations appear in celestial mechanics (planets, stars and galaxies); numerical linear algebra is important for data analysis; stochastic differential equations and Markov chains are essential in simulating living cells for medicine and biology.

Before the advent of modern computers numerical methods often depended on hand interpolation in large printed tables. Since the mid 20th century, computers calculate the required functions instead. These same interpolation formulas nevertheless continue to be used as part of the software algorithms for solving differential equations.

General introduction

The overall goal of the field of numerical analysis is the design and analysis of techniques to give approximate but accurate solutions to hard problems, the variety of which is suggested by the following:

- Advanced numerical methods are essential in making numerical weather prediction feasible.
- Computing the trajectory of a spacecraft requires the accurate numerical solution of a system of ordinary differential equations.
- Car companies can improve the crash safety of their vehicles by using computer simulations of car crashes. Such simulations essentially consist of solving partial differential equations numerically.
- Hedge funds (private investment funds) use tools from all fields of numerical analysis to attempt to calculate the value of stocks and derivatives more precisely than other market participants.
- Airlines use sophisticated optimization algorithms to decide ticket prices, airplane and crew assignments and fuel needs. Historically, such algorithms were developed within the overlapping field of operations research.
- Insurance companies use numerical programs for actuarial analysis.

Approximation theory

Use computable functions $p(x)$ to approximate the values of functions $f(x)$ that are not easily computable or use approximations to simplify dealing with such functions. The most popular types of computable functions $p(x)$ are polynomials, rational functions, and piecewise versions of them, for example spline functions. Trigonometric polynomials are also a very useful choice.

- *Best approximations*. Here a given function $f(x)$ is approximated within a given finite-dimensional family of computable functions. The quality of the approximation is expressed by a functional, usually the maximum absolute value of the approximation error or an integral involving the error. Least squares approximations and minimax approximations are the most popular choices.

- *Interpolation*. A computable function $p(x)$ is to be chosen to agree with a given $f(x)$ at a given finite set of points x . The study of determining and analyzing such interpolation functions is still an active area of research, particularly when $p(x)$ is a multivariate polynomial.
- *Fourier series*. A function $f(x)$ is decomposed into orthogonal components based on a given orthogonal basis $\{\varphi1,\varphi2,\ldots\}$, and then $f(x)$ is approximated by using only the largest of such components. The convergence of Fourier series is a classical area of mathematics, and it is very important in many fields of application. The development of the Fast Fourier Transform in 1965 spawned a rapid progress in digital technology. In the 1990s wavelets became an important tool in this area.
- *Numerical integration and differentiation*. Most integrals cannot be evaluated directly in terms of elementary functions, and instead they must be approximated numerically. Most functions can be differentiated analytically, but there is still a need for numerical differentiation, both to approximate the derivative of numerical data and to obtain approximations for discretizing differential equations.

Numerical methods

Mathematical techniques for solving practical problems. They are called **numerical methods** because they produce the solution as real numbers such as " 3.1768 ", rather than as algebraic expressions (such as " $x^2 + c$ ") or surds (such as " 2 ± sqrt(11) " where sqrt means the square root function).

Numerical methods are, some might say, a lot easier to use than the algebraic methods.

Numerical methods to solve three types of problem:

- Find a solution to a nonlinear equation $F(x) = 0$
- Find an interpolated value from a table of data
- Find the numerical solution of a differential equation

Direct and iterative methods

Direct methods compute the solution to a problem in a finite number of steps. These methods would give the precise answer if they were performed in infinite precision arithmetic. Examples include Gaussian elimination, the QR factorization method for solving systems of linear equations, and the simplex method of linear programming. In practice, finite precision is used and the result is an approximation of the true solution (assuming stability).

In contrast to direct methods, iterative methods are not expected to terminate in a finite number of steps. Starting from an initial guess, iterative methods form successive approximations that converge to the exact solution only in the limit. A convergence test, often involving the residual, is specified in order to decide when a sufficiently accurate solution has (hopefully) been found. Even using infinite precision arithmetic these methods would not reach the solution within a finite number of steps (in general). Examples include Newton's method, the bisection method, and Jacobi iteration. In computational matrix algebra, iterative methods are generally needed for large problems.

Iterative methods are more common than direct methods in numerical analysis. Some methods are direct in principle but are usually used as though they were not, e.g. GMRES and the conjugate gradient method. For these methods the number of steps needed to obtain the exact solution is so large that an approximation is accepted in the same manner as for an iterative method.

Direct vs iterative methods

Consider the problem of solving

$$3x^3 + 4 = 28$$

for the unknown quantity x.

Direct method

	$3x^3 + 4 = 28.$
Subtract 4	$3x^3 = 24.$
Divide by 3	$x^3 = 8.$
Take cube roots	$x = 2.$

For the iterative method, apply the bisection method to $f(x) = 3x^3 - 24$. The initial values are $a = 0$, $b = 3$, $f(a) = -24$, $f(b) = 57$.

Iterative method

a	***b***	**mid**	***f*(mid)**
0	3	1.5	−13.875
1.5	3	2.25	10.17...
1.5	2.25	1.875	−4.22...
1.875	2.25	2.0625	2.32...

We conclude from this table that the solution is between 1.875 and 2.0625. The algorithm might return any number in that range with an error less than 0.2.

Discretization

Furthermore, continuous problems must sometimes be replaced by a discrete problem whose solution is known to approximate that of the continuous problem; this process is called *discretization*. For example, the solution of a differential equation is a function. This function must be represented by a finite amount of data, for instance by its value at a finite number of points at its domain, even though this domain is a continuum.

Discretization and numerical integration

In a two-hour race, we have measured the speed of the car at three instants and recorded them in the following table.

Time	0:20	1:00	1:40
km/h	140	150	180

A **discretization** would be to say that the speed of the car was constant from 0:00 to 0:40, then from 0:40 to 1:20 and finally from 1:20 to 2:00. For instance, the total distance traveled in the first 40 minutes is approximately (2/3 h × 140 km/h) = 93.3 km. This would allow us to estimate the total distance traveled as 93.3 km + 100 km + 120 km = 313.3 km, which is an example of **numerical integration** (see below) using a Riemann sum, because displacement is the integral of velocity.

Ill-conditioned problem: Take the function $f(x) = 1/(x - 1)$. Note that $f(1.1) = 10$ and $f(1.001) = 1000$: a change in x of less than 0.1 turns into a change in $f(x)$ of nearly 1000. Evaluating $f(x)$ near $x = 1$ is an ill-conditioned problem.

Well-conditioned problem: By contrast, evaluating the same function $f(x) = 1/(x - 1)$ near $x = 10$ is a well-conditioned problem. For instance, $f(10) = 1/9 \approx 0.111$ and $f(11) = 0.1$: a modest change in x leads to a modest change in $f(x)$.

Generation and propagation of errors

The study of errors forms an important part of numerical analysis. There are several ways in which error can be introduced in the solution of the problem.

Round-off

Round-off errors arise because it is impossible to represent all real numbers exactly on a machine with finite memory (which is what all practical digital computers are).

Truncation and discretization error

Truncation errors are committed when an iterative method is terminated or a mathematical procedure is approximated, and the approximate solution differs from the exact solution. Similarly, discretization induces a discretization error because the solution of the discrete problem does not coincide with the solution of the continuous problem. For instance, in the iteration in the sidebar to compute the solution of $3x^3+4=28$, after 10 or so iterations, we conclude that the root is roughly 1.99 (for example). We therefore have a truncation error of 0.01.

Once an error is generated, it will generally propagate through the calculation. For instance, we have already noted that the operation + on a calculator (or a computer) is inexact. It follows that a calculation of the type $a+b+c+d+e$ is even more inexact.

What does it mean when we say that the truncation error is created when we approximate a mathematical procedure? We know that to integrate a function exactly requires one to find the sum of infinite trapezoids. But numerically one can find the sum of only finite trapezoids, and hence the approximation of the mathematical procedure. Similarly, to differentiate a function, the differential element approaches zero but numerically we can only choose a finite value of the differential element.

Areas of study

The field of numerical analysis includes many sub-disciplines. Some of the major ones are:

Computing values of functions

One of the simplest problems is the evaluation of a function at a given point. The most straightforward approach, of just plugging in the number in the formula is sometimes not very efficient. For polynomials, a better approach is using the Horner scheme, since it reduces the necessary number of multiplications and additions. Generally, it is important to estimate and control round-off errors arising from the use of floating point arithmetic.

Interpolation, extrapolation, and regression

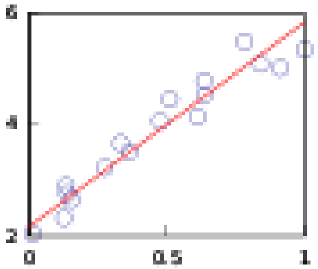
Interpolation solves the following problem: given the value of some unknown function at a number of points, what value does that function have at some other point between the given points?

Extrapolation is very similar to interpolation, except that now we want to find the value of the unknown function at a point which is outside the given points.

Regression is also similar, but it takes into account that the data is imprecise. Given some points, and a measurement of the value of some function at these points (with an error), we want to determine the unknown function. The least squares-method is one popular way to achieve this.

Interpolation: We have observed the temperature to vary from 20 degrees Celsius at 1:00 to 14 degrees at 3:00. A linear interpolation of this data would conclude that it was 17 degrees at 2:00 and 18.5 degrees at 1:30pm.

Extrapolation: If the gross domestic product of a country has been growing an average of 5% per year and was 100 billion dollars last year, we might extrapolate that it will be 105 billion dollars this year.



Regression: In linear regression, given *n* points, we compute a line that passes as close as possible to those *n* points.

Optimization: Say you sell lemonade at a *lemonade stand*, and notice that at $1, you can sell 197 glasses of lemonade per day, and that for each increase of $0.01, you will sell one glass of lemonade less per day. If you could charge $1.485, you would maximize your profit, but due to the constraint of having to charge a whole cent amount, charging $1.48 or $1.49 per glass will both yield the maximum income of $220.52 per day.

Differential equation: If you set up 100 fans to blow air from one end of the room to the other and then you drop a feather into the wind, what happens? The feather will follow the air currents, which may be very complex. One approximation is to measure the speed at which the air is blowing near the feather every second, and advance the simulated feather as if it were moving in a straight line at that same speed for one second, before measuring the wind speed again. This is called the Euler method for solving an ordinary differential equation.

Solving equations and systems of equations

Another fundamental problem is computing the solution of some given equation. Two cases are commonly distinguished, depending on whether the equation is linear or not. For instance, the equation $2X+5=3$ is linear while $2X2+5=3$ is not.

Much effort has been put in the development of methods for solving systems of linear equations. Standard direct methods, i.e., methods that use some matrix decomposition are Gaussian elimination, LU decomposition, Cholesky decomposition for symmetric (or hermitian) and positive-definite matrix, and QR decomposition for non-square matrices. Iterative methods such as the Jacobi method, Gauss–Seidel method, successive over-relaxation and conjugate gradient method are usually preferred for large systems. General iterative methods can be developed using a matrix splitting.

Root-finding algorithms are used to solve nonlinear equations (they are so named since a root of a function is an argument for which the function yields zero). If the function is differentiable and the derivative is known, then Newton's method is a popular choice. Linearization is another technique for solving nonlinear equations.

Solving eigenvalue or singular value problems

Several important problems can be phrased in terms of eigenvalue decompositions or singular value decompositions. For instance, the spectral image compression algorithm is based on the singular value decomposition. The corresponding tool in statistics is called principal component analysis.

Optimization

Optimization problems ask for the point at which a given function is maximized (or minimized). Often, the point also has to satisfy some constraints.

The field of optimization is further split in several subfields, depending on the form of the objective function and the constraint. For instance, linear programming deals with the case that both the objective function and the constraints are linear. A famous method in linear programming is the simplex method.

The method of Lagrange multipliers can be used to reduce optimization problems with constraints to unconstrained optimization problems.

Evaluating integrals

Numerical integration, in some instances also known as numerical quadrature, asks for the value of a definite integral. Popular methods use one of the Newton–Cotes formulas (like the midpoint rule or Simpson's rule) or Gaussian quadrature. These methods rely on a "divide and conquer" strategy, whereby an integral on a relatively large set is broken down into integrals on smaller sets. In higher dimensions, where these methods become prohibitively expensive in terms of computational effort, one may use Monte Carlo or quasi-Monte Carlo methods , or in modestly large dimensions, the method of sparse grids.

Differential equations

Numerical analysis is also concerned with computing (in an approximate way) the solution of differential equations, both ordinary differential equations and partial differential equations.

Partial differential equations are solved by first discretizing the equation, bringing it into a finite-dimensional subspace. This can be done by a finite element method, a finite difference method, or (particularly in engineering) a finite volume method. The theoretical justification of these methods often involves theorems from functional analysis. This reduces the problem to the solution of an algebraic equation.

Software

Since the late twentieth century, most algorithms are implemented in a variety of programming languages. The Netlib repository contains various collections of software routines for numerical problems, mostly in Fortran and C. Commercial products implementing many different numerical algorithms include the IMSL and NAG libraries; a free-software alternative is the GNU Scientific Library.

There are several popular numerical computing applications such as MATLAB, TK Solver, S-PLUS, and IDL as well as free and open source alternatives such as FreeMat, Scilab, GNU Octave (similar to Matlab), and IT++ (a C++ library). There are also programming languages such as R (similar to S-PLUS) and Python with libraries such as NumPy, SciPy and SymPy. Performance varies widely: while vector and matrix operations are usually fast, scalar loops may vary in speed by more than an order of magnitude.

Many computer algebra systems such as Mathematica also benefit from the availability of arbitrary precision arithmetic which can provide more accurate results.

Also, any spreadsheet software can be used to solve simple problems relating to numerical analysis.

CHAPTER 11: INTRODUCTION TO MODELING

Introduction to Mathematical modeling

A **mathematical model** is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed **mathematical modeling**. Mathematical models are used in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines (such as computer science, artificial intelligence), as well as in the social sciences (such as economics, psychology, sociology, political science).

A model may help to explain a system and to study the effects of different components, and to make predictions about behavior.

Hypothesis, Model, Theory & Law

In common usage, the words hypothesis, model, theory, and law have different interpretations and are at times used without precision, but in science they have very exact meanings.

Hypothesis

It is a **limited statement** regarding the cause and effect in a specific situation, which can be tested by experimentation and observation or by statistical analysis of the probabilities from the data obtained. The outcome of the test hypothesis should be currently unknown, so that the results can provide useful data regarding the validity of the hypothesis.

Sometimes a hypothesis is developed that must wait for new knowledge or technology to be testable. The concept of atoms was proposed by the ancient Greeks, who had no means of testing it. Centuries later, when more knowledge became available, the hypothesis gained support and was eventually accepted by the scientific community, though it has had to be amended many times over the year. Atoms are not indivisible, as the Greeks supposed.

Model

A *model* is used for situations when it is known that the hypothesis has a limitation on its validity.

Several classes of models

Three classes appear :

- Models which come from laws of physics: this is the case for gravitation laws, Maxwell equations (waves), Navier-Stokes equations (fluid dynamics), and so on;
- Model which come from empirical laws, such as air resistance for a movement: this laws are of empirical nature;
- Models which use statistical laws, for instance that fit a line between several points and assume the response to be linear.

Theory & Law

A *scientific theory* or *law* represents a hypothesis (or group of related hypotheses) which has been confirmed through repeated testing, almost always conducted over a span of many years.

Generally, a theory is an explanation for a set of related phenomena, like the theory of evolution or the big bang theory.

The word "law" is often invoked in reference to a specific mathematical equation that relates the different elements within a theory. Pascal's Law refers an equation that describes differences in pressure based on height. In the overall theory of universal gravitation developed by Sir Isaac Newton, the key equation that describes the gravitational attraction between two objects is called the law of gravity.

These days, physicists rarely apply the word "law" to their ideas. In part, this is because so many of the previous "laws of nature" were found to be not so much laws as guidelines, that work well within certain parameters but not within others.

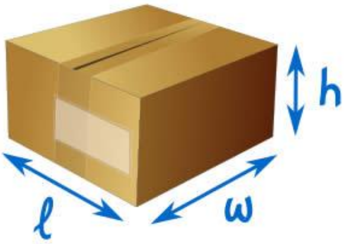
General rules of mathematical modeling

- Look at how others model similar situations; adapt their models to the present situation.
- Collect/ask for background information needed to understand the problem.
- Start with simple models; add details as they become known and useful or necessary.
- Find all relevant quantities and make them precise.
- Find all relevant relationships between quantities ([differential] equations, inequalities, case distinctions).
- Locate/collect/select the data needed to specify these relationships.
- Find all restrictions that the quantities must obey (sign, limits, forbidden overlaps, etc.). Which restrictions are hard, which soft? How soft?
- Try to incorporate qualitative constraints that rule out otherwise feasible results (usually from inadequate previous versions).
- Find all goals (including conflicting ones)
- Play the devil's advocate to find out and formulate the weak spots of your model.
- Sort available information by the degree of impact expected/hoped for.
- Create a hierarchy of models: from coarse, highly simplifying models to models with all known details. Are there useful toy models with simpler data? Are there limiting cases where the model simplifies? Are there interesting extreme cases that help discover difficulties?
- First solve the coarser models (cheap but inaccurate) to get good starting points for the finer models (expensive to solve but realistic)
- Try to have a simple working model (with report) after 1/3 of the total time planned for the task. Use the remaining time for improving or expanding the model based on your experience, for making the programs more versatile and speeding them up, for polishing documentation, etc.
- Good communication is essential for good applied work.
- The responsibility for understanding, for asking the questions that lead to it, for recognizing misunderstanding (mismatch between answers expected and answers received), and for overcoming them lies with the mathematician. You cannot usually assume your customer to understand your scientific jargon.
- Be not discouraged. Failures inform you about important missing details in your understanding of the problem (or the customer/boss) - utilize this information!
- There are rarely perfect solutions. Modeling is the art of finding a satisfying compromise. Start with the highest standards, and lower them as the deadline approaches. If you have results early, raise your standards again.
- Finish your work in time.

Construct generalized models

Mathematics can be used to "model", or represent, how the real world works.

Example: how much space is inside this cardboard box?



We know three measurements:

- **l** (length),
- **w** (width), and
- **h** (height),

and the formula for the volume of a cuboid is:

$$\text{Volume} = l \times w \times h$$

So we have a (very simple) mathematical model of the space in that box.

Accurate?

The model is not the same as the real thing.

In our example we did not think about the thickness of the cardboard, or many other "real world" things.

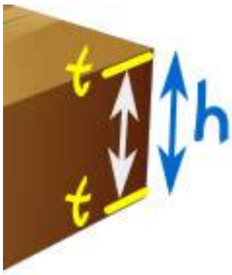
But hopefully it is **good enough to be useful**.

> If we are charged by the volume of the box we send, we can take a few measurements and know how much to pay.
>
> It can also be useful when deciding which box to buy when we need to pack things.

So the model is useful!

But maybe we need more accuracy, we might need to send hundreds of boxes every day, and the thickness of the cardboard matters. So let's see if we can **improve the model**:

The cardboard is "t" thick, and all measurements are outside the box ... how much space is inside?



The inside measurements need to be reduced by the thickness of each side:

- The inside length is **l-2t**
- The inside width is **w-2t**,
- The inside height is **h-2t**

and now the formula is:

Inside Volume = (l-2t) × (w-2t) × (h-2t)

Now we have a **better** model. Still not perfect (did we consider wasted space because we could not pack things neatly, etc ...), but better.

So a model is not reality, but should be good enough to be useful.

Playing With The Model

Now we have a model, we can use it in different ways:

Example: Your company uses 200x300x400 mm size boxes, and the cardboard is 5mm thick.

Someone suggests using 4mm cardboard ... how much better is that?

Let us compare the two volumes:

- Current Volume = (200-2×5) × (300-2×5) × (400-2×5) = 21,489,000 mm^3
- New Volume = (200-2×4) × (300-2×4) × (400-2×4) = 21,977,088 mm^3

That is a change of:

(21,977,088-21,489,000)/21,489,000 ≈ 2% more volume

So the model is **useful**. It lets us know we will get 2% more space inside the box (for the same outside measurements).

But there are still "real world" things to think about, such as "will it be strong enough?"

Thinking Clearly

To set up a mathematical model we also need to think clearly about the facts!

Example: on our street there are twice as many dogs as cats. How do we write this as an equation?

- Let D = number of dogs
- Let C = number of cats

Now ... is that: 2D = C or should it be: D = 2C

Think carefully now!

The correct answer is D = 2C

(**2D = C** is a common mistake, as the question is written "twice ... dogs ... cats")

Here is another one:

Example: You are the supervisor of 8-hour shift workers. They recently had their break times reduced by 10 minutes but total production did not improve.

At first glance there is nothing to model, because there was no change in production.

But wait a minute ... they are working 10 minutes more, but producing the same amount, so **production per hour** must have dropped!

Let us assume they used to work 7 hours (420 minutes):

Change in production per hour = 410/420 = **0.976...**

Which is a **reduction of more than 2%**

But even worse: the first few hours of the shift are not be affected by the shorter break time, so it could be a 4 or 5% reduction later in the shift.

You could recommend:

- looking at production rates for every hour of the shift
- trying different break times to see how they affect production

Steps into Model Building

A Bigger Example: Most Economical Size, OK, let us have a go at building and using a mathematical model to solve a real world question.

Your company is going to make its own boxes!

It has been decided the box should hold **0.02m^3** (0.02 cubic meters which is equal to 20 liters) of nuts and bolts.

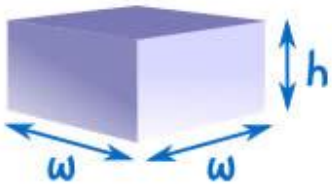
The box should have a square base, and double thickness top and bottom.

Cardboard costs **$0.30** per square meter.

It is up to you to decide the most economical size.

Step One: Draw a sketch!

It helps to sketch out what we are trying to solve!



The base is square, so we will just use "w" for both lengths

The box has 4 sides, and double tops and bottoms.

The box shape could be cut out like this (but is probably more complicated):

Step Two: Make Formulas.

Ignoring thickness for this model:

$$\text{Volume} = w \times w \times h = w^2h$$

And we are told that the volume should be 0.02m^3:

$$w^2h = 0.02$$

Areas:

$$\text{Area of the 4 Sides} = 4 \times w \times h = 4wh$$

$$\text{Area of Double Tops and Bases} = 4 \times w \times w = 4w^2$$

Total cardboard needed:

$$\text{Area of Cardboard} = 4wh + 4w^2$$

Step Three: Make a Single Formula For Cost

We want a single formula for cost:

$$\text{Cost} = \$0.30 \times \text{Area of Cardboard}$$

$$= \$0.30 \times (4wh + 4w^2)$$

And that is the cost when we know width **and** height.

That could be hard to work with ... a function with two variables.

But we can make it simpler! Because width and height are already related by the volume:

$$\text{Volume} = w^2h = 0.02$$

... which can be rearranged to ...

$$h = 0.02/w^2$$

... and that can be put into the cost formula ...

$$\text{Cost} = \$0.30 \times (4w\times\mathbf{0.02/w^2} + 4w^2)$$

And now the cost is related directly to **width only**.

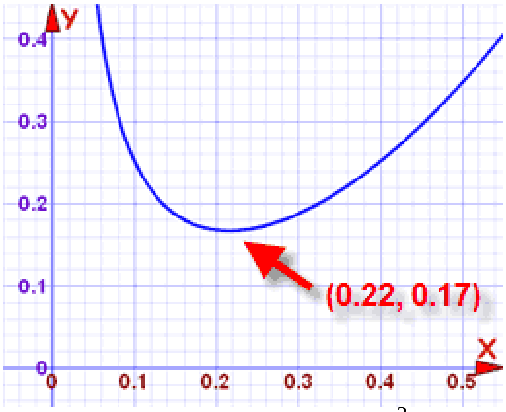
With a little simplification we get:

$$\text{Cost} = \$0.30 \times (0.08/w + 4w^2)$$

Step Four: Plot it and find minimum cost

What to plot? Well, the formula only makes sense for widths greater than zero, and I also found that for widths above 0.5 the cost just gets bigger and bigger.

So here is a plot of that cost formula for **widths between 0.0 m and 0.55 m**:



Plot of $y = 0.3(0.08/x+4x^2)$
x is width, and **y** is cost

Just by eye, I see the cost reaches a minimum at about **(0.22, 0.17)**. In other words:

- when the width is about **0.22 m** (x-value),
- the minimum cost is about **$0.17** per box (y-value).

In fact, looking at the graph, the width could be anywhere between 0.20 and 0.24 without affecting the minimum cost very much.

Step Five: Recommendations

Using this mathematical model you can now recommend:

- Width = 0.22 m
- Height = $0.02/w^2 = 0.02/0.22^2 = 0.413$ m
- Cost = $\$0.30 \times (0.08/w+ 4w^2) = \$0.30 \times (0.08/0.22+ 4\times0.22^2) = \$0.167$

Or about 16.7 cents per box

But any width between 0.20 m and 0.24 m is fine.

You might also like to suggest improvements to this model:

- Include cost of glue/staples and assembly
- Include wastage when cutting box shape from cardboard.
- Is this box a good shape for packing, handling and storing?
- Any other ideas you may have!

Predicting the Future

Mathematical models can also be used to forecast future behavior.

Example: An ice cream company keeps track of how many ice creams get sold on different days.

By comparing this to the weather on each day they can make a mathematical model of **sales versus weather**.

They can then predict future sales based on the weather forecast, and decide how many ice creams they need to make ... ahead of time!

Computer Modeling

Mathematical models can get very complex, and so the mathematical rules are often written into computer programs, to make a computer model.

Examples include:

- Weather prediction
- Economic Models (predicting interest rates, unemployment, etc)
- Models of how large structures behave under stress (bridges, skyscrapers, etc)
- Many more ...

If you become an expert in any of those you will have a job for life!

Meaning of mathematical Model pseudo-code

Pseudo-code is an informal way of programming description that does not require any strict programming language syntax or underlying technology considerations. It is used for creating an outline or a rough draft of a program. Pseudo-code summarizes a program's flow, but excludes underlying details. System designers write pseudo-code to ensure that programmers understand a software project's requirements and align code accordingly. **Pseudo-code** is an informal high-level description of the operating principle of a computer program or other **algorithm**.

An algorithm is a procedure for solving a problem in terms of the steps to be executed and the order in which those steps are to be executed. An algorithm is merely the sequence of steps taken to solve a mathematical problem and may be Regarded as a pseudo-code

Logical models, Statistical models and Other Models

Logic model

A **logic model** (also known as a logical framework, theory of change, or program matrix) is a tool used by funders, managers, and evaluators of programs to evaluate the effectiveness of a program. They can also be used during planning and implementation. Logic models are usually a graphical depiction of the logical relationships between the resources, activities, outputs and outcomes of a program. While there are many ways in which logic models can be presented, the underlying purpose of constructing a logic model is to assess the "if-then" (causal) relationships between the elements of the program.

In its simplest form, a logic model has four components:

Inputs	Activities	Outputs	Outcomes/impacts
what resources go into a program	*what activities the program undertakes*	*what is produced through those activities*	*the changes or benefits that result from the program*
e.g. money, staff, equipment	e.g. development of materials, training programs	e.g. number of booklets produced, workshops held, people trained	e.g. increased skills/ knowledge/ confidence, leading in longer-term to promotion, new job, etc.

Following the early development of the logic model in the 1970s by Carol Weiss, Joseph Wholey and others, many refinements and variations have been added to the basic concept. Many versions of logic models set out a series of outcomes/impacts, explaining in more detail the logic of how an intervention contributes to intended or observed results. This will often include distinguishing between short-term, medium-term and long-term results, and between direct and indirect results.

Some logic models also include assumptions, which are beliefs the prospective grantees have about the program, the people involved, and the context and the way the prospective grantees think the program will work, and external factors, consisting of the environment in which the program exists, including a variety of external factors that interact with and influence the program action.

University Cooperative Extension Programs have developed a more elaborate logic model, called the Program Action Logic Model, which includes six steps:

- **Inputs** (what we invest)
- **Outputs:**
    - **Activities** (the actual tasks we do)
    - **Participation** (who we serve; customers & stakeholders)
    - **Engagement** (how those we serve engage with the activities)
- **Outcomes/Impacts:**
    - **Short Term** (learning: awareness, knowledge, skills, motivations)
    - **Medium Term** (action: behavior, practice, decisions, policies)
    - **Long Term** (consequences: social, economic, environmental etc.)

Advantages

By describing work in this way, managers have an easier way to define the work and measure it. Performance measures can be drawn from any of the steps. One of the key insights of the logic model is the importance of measuring final outcomes or results, because it is quite possible to waste time and money (inputs), "spin the wheels" on work activities, or produce outputs without achieving desired outcomes. It is these outcomes (impacts, long-term results) that are the only justification for doing the work in the first place. For commercial organizations, outcomes relate to profit. For not-for-profit or governmental organizations, outcomes relate to successful achievement of mission or program goals.

Uses of the logic model

Program planning - helps managers to 'plan with the end in', rather than just consider inputs (e.g. budgets, employees) or just the tasks that must be done.

Performance evaluation - used in government or not-for-profit organizations, where the mission and vision are not aimed at achieving a financial benefit. In such situations, where profit is not the intended result, it may be difficult to monitor progress toward outcomes. A program logic model provides such indicators, in terms of output and outcome measures of performance.

The logic model and other management frameworks

There are numerous other popular management frameworks that have been developed in recent decades. This often causes confusion, because the various frameworks have different functions. It is important to select the *right tool for the job*. The following list of popular management tools is suggested to indicate where they are most appropriate (this list is by no means complete).

Organizational assessment tools: Fact-gathering tools for a comprehensive view of the as-is situation in an organization, but without prescribing how to change it:

Strategic planning tools: For identifying and prioritizing major long-term desired results in an organization, and strategies to achieve those results:

Program planning and evaluation tools: For developing details of individual programs (what to do and what to measure) once overall strategies have been defined:

Performance measurement tools: For measuring, monitoring and reporting the quality, efficiency, speed, cost and other aspects of projects, programs and/or processes:

Process improvement tools: For monitoring and improving the quality or efficiency of work processes:

Process standardization tools: For maintaining and documenting processes or resources to keep them repeatable and stable:

Statistical models

A **statistical model** is a class of mathematical model, which embodies a set of assumptions concerning the generation of some sample data, and similar data from a larger population. A statistical model represents, often in considerably idealized form, the data-generating process.

The assumptions embodied by a statistical model describe a set of probability distributions, some of which are assumed to adequately approximate the distribution from which a particular data set is sampled. The probability distributions inherent in statistical models are what distinguishes statistical models from other, non-statistical, mathematical models.

A statistical model is usually specified by mathematical equations that relate one or more random variables and possibly other non-random variables. As such, "a model is a formal representation of a theory" (Herman Adèr quoting Kenneth Bollen).

All statistical hypothesis tests and all statistical estimators are derived from statistical models. More generally, statistical models are part of the foundation of statistical inference.

Formal definition

In mathematical terms, a statistical model is usually thought of as a pair ($\mathcal{S},\mathcal{P}$), where $\mathcal{S}$ is the set of possible observations, i.e. the sample space, and $\mathcal{P}$ is a set of probability distributions on $\mathcal{S}$.

The intuition behind this definition is as follows. It is assumed that there is a "true" probability distribution induced by the process that generates the observed data. We choose to represent a set (of distributions) which contains a distribution that adequately approximates the true distribution. Note that we do not require that contains the true distribution, and in practice that is rarely the case. Indeed, as Burnham & Anderson state, "A model is a simplification or approximation of reality and hence will not reflect all of reality"—whence the saying "all models are wrong".

The set $\mathcal{P}$ is almost always parameterized: $\mathcal{P} = \{\mathcal{P}_\Theta : \Theta \in \Theta\}$. The set Θ defines the parameters of the model. A parameterization is generally required to have distinct parameter values give rise to distinct distributions, i.e. $\mathcal{P}_{\Theta 1} = \mathcal{P}_{\Theta 2} \Rightarrow \Theta_1 = \Theta_2$. Must hold (in other words, it must be injective). A parameterization that meets the condition is said to be *identifiable*.

An example

Height and age are each probabilistically distributed over humans. They are stochastically related: when we know that a person is of age 10, this influences the chance of the person being 5 feet tall. We could formalize that relationship in a linear regression model with the following form: $\text{height}_i = b_0 + b_1\text{age}_i + \varepsilon_i$, where b_0 is the intercept, b_1 is a parameter that age is multiplied by to get a prediction of height, ε is the error term, and i identifies the person. This implies that height is predicted by age, with some error.

An admissible model must be consistent with all the data points. Thus, the straight line ($\text{height}_i = b_0 + b_1\text{age}_i$) is *not* a model of the data. The line cannot be a model, unless it exactly fits all the data points—i.e. all the data points lie perfectly on a straight line. The error term, ε_i, must be included in the model, so that the model is consistent with all the data points.

To do statistical inference, we would first need to assume some probability distributions for the ε_i. For instance, we might assume that the ε_i distributions are i.i.d. Gaussian, with zero mean. In this instance, the model would have 3 parameters: b_0, b_1, and the variance of the Gaussian distribution.

We can formally specify the model in the form ($\mathcal{S}$, $\mathcal{P}$) as follows. The sample space, $\mathcal{S}$, of our model comprises the set of all possible pairs (age, height). Each possible value of $\Theta = (b_0, b_1, \sigma^2)$ determines a distribution on $\mathcal{S}$; denote that distribution by $\mathcal{P}_\Theta$. If Θ is the set of all possible values of Θ, then $\mathcal{P} = \{\mathcal{P}_\Theta : \Theta \in \Theta\}$.. (The parameterization is identifiable, and this is easy to check.)

In this example, the model is determined by (1) specifying $\mathcal{S}$ and (2) making some assumptions relevant to $\mathcal{P}$. There are two assumptions: that height can be approximated by a linear function of age; that errors in the approximation are distributed as i.i.d. Gaussian. The assumptions are sufficient to specify $\mathcal{P}$—as they are required to do.

General remarks

A statistical model is a special class of mathematical model. What distinguishes a statistical model from other mathematical models is that a statistical model is non-deterministic. Thus, in a statistical model specified via mathematical equations, some of the variables do not have specific values, but instead have probability distributions; i.e. some of the variables are stochastic. In the example above, ε is a stochastic variable; without that variable, the model would be deterministic.

Statistical models are often used even when the physical process being modeled is deterministic. For instance, coin tossing is, in principle, a deterministic process; yet it is commonly modeled as stochastic (via a Bernoulli process).

There are three purposes for a statistical model, according to Konishi & Kitagawa.

- Predictions
- Extraction of information
- Description of stochastic structures

Graphical Representation of statistical Data

Statistics is a special subject that deals with large (usually) numerical data. The statistical data can be represented graphically. In fact, the graphical representation of statistical data is an essential step during statistical analysis.
Statistical surveys and experiments provides valuable information about numerical scores. For better understanding and making conclusions and interpretations, the data should be managed and organized in a systematic form. A graph is the representation of data by using graphical symbols such as lines, bars, pie slices, dots etc. A graph does represent a numerical data in the form of a qualitative structure and provides important information.

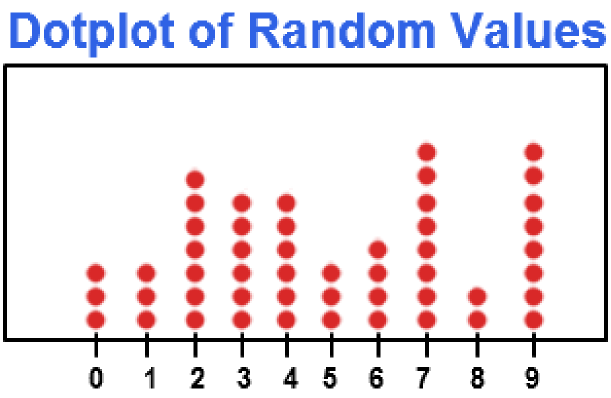
Let us go ahead and study about various types of graphical representations of the data.

Dot Plots

The dot plot is one of the most simplest ways of graphical representation of the statistical data. As the name itself suggests, a dot plot uses the dots. It is a graphic display which usually compares frequency within different categories.

The dot plot is composed of dots that are to be plotted on a graph paper.

A dot plot may look like:



In the dot plot, every dot denotes a specific number of observations belonging to a data set. One dot usually represents one observation.

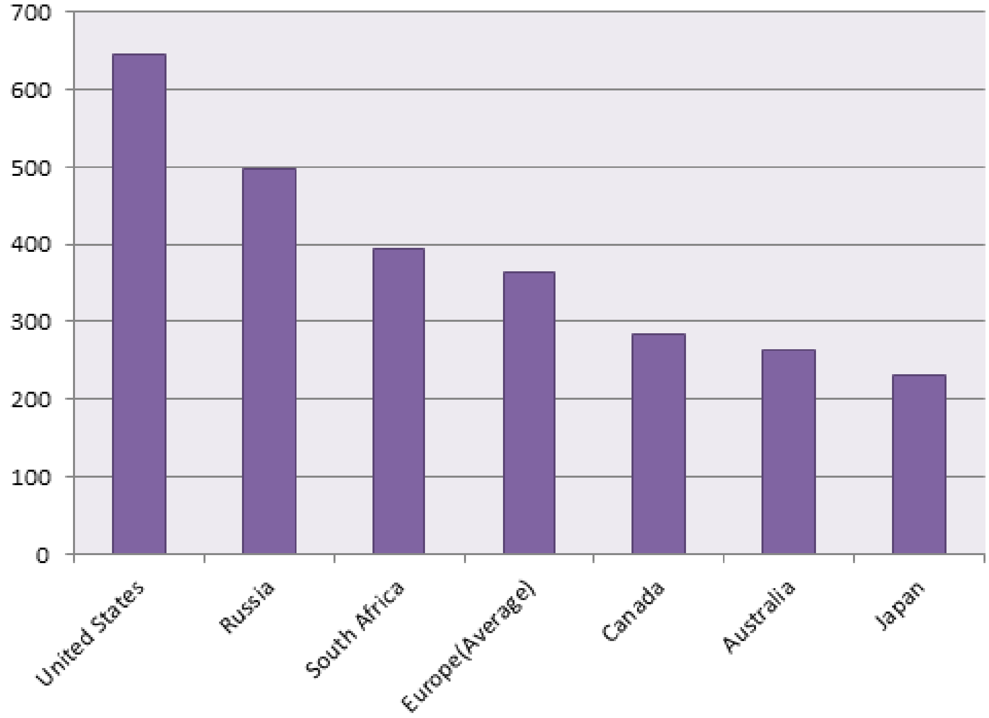
These dots are to be marked in the form of a column for each category. In this way, the height of each column shows the corresponding frequency of some category.
The dot plots are quite useful when there are small amount of data is given within the small number of categories.

Bar Graph

A bar graph is a very frequently used graph in statistics as well as in media. A bar graph is a type of graph which contains rectangles or rectangular bars. The lengths of these bars should be proportional to the numerical values represented by them. In bar graph, the bars may be plotted either horizontally or vertically. But a vertical bar graph (also known as column bar graph) is used more than a horizontal one.

A vertical bar graph is shown below:

Number of students went to different states for study:



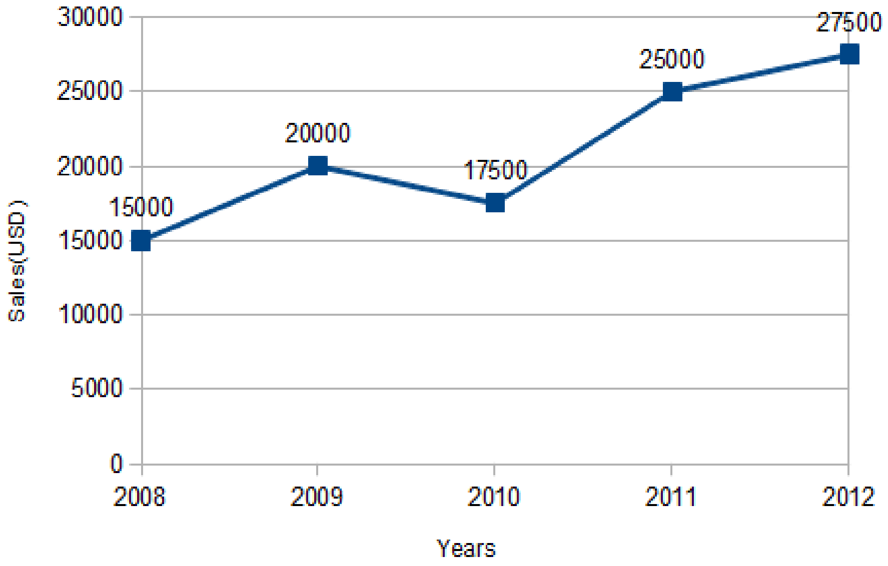
The rectangular bars are separated by some distance in order to distinguish them from one another. The bar graph shows comparison among the given categories.

Mostly, horizontal axis of the graph represents specific categories and vertical axis shows the discrete numerical values.

Line Graph

A line graph is a kind of graph which represents data in a way that a series of points are to be connected by segments of straight lines. In a line graph, the data points are plotted on a graph and they are joined together with straight line.

A sample line graph is illustrated in the following diagram:

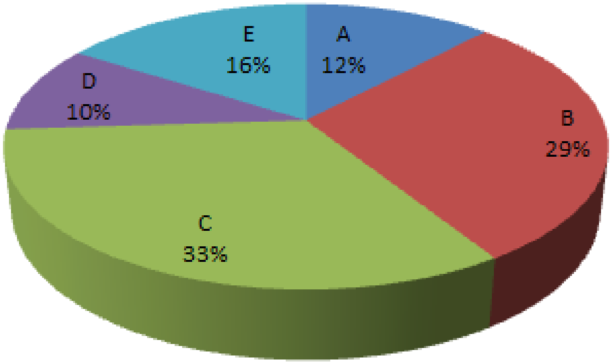


The line graphs are used in the science, statistics and media. Line graphs are very easy to create. These are quite popular in comparison with other graphs since they visualize characteristics revealing data trends very clearly. A line graph gives a clear visual comparison between two variables which are represented on X-axis and Y-axis.

Circle Graph

A circle graph is also known as a pie graph or pie chart. It is called so since it is similar to slice of a "pie". A pie graph is defined as a graph which contains a circle which is divided into sectors. These sectors illustrate the numerical proportion of the data.

A pie chart are shown in the following diagram:

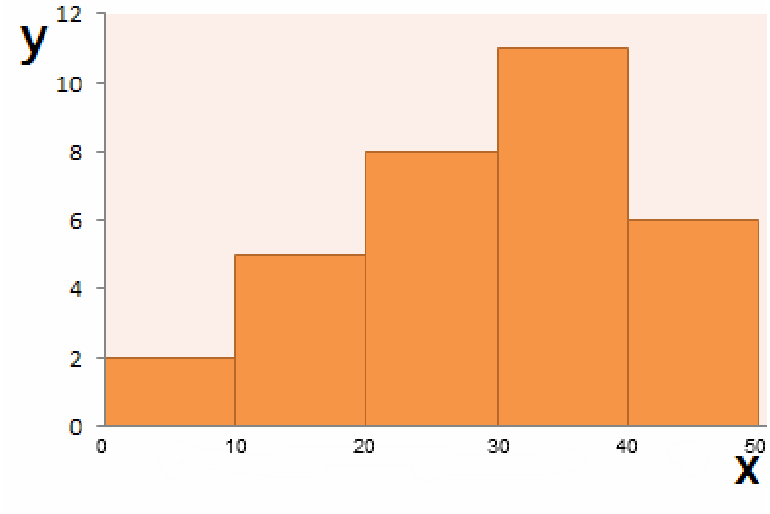


The arc lengths of the sectors, in pie chart, are proportional to the numerical value they represent. Circle graphs are quite commonly seen in mass media as well as in business world.

Histogram and Frequency Polygon

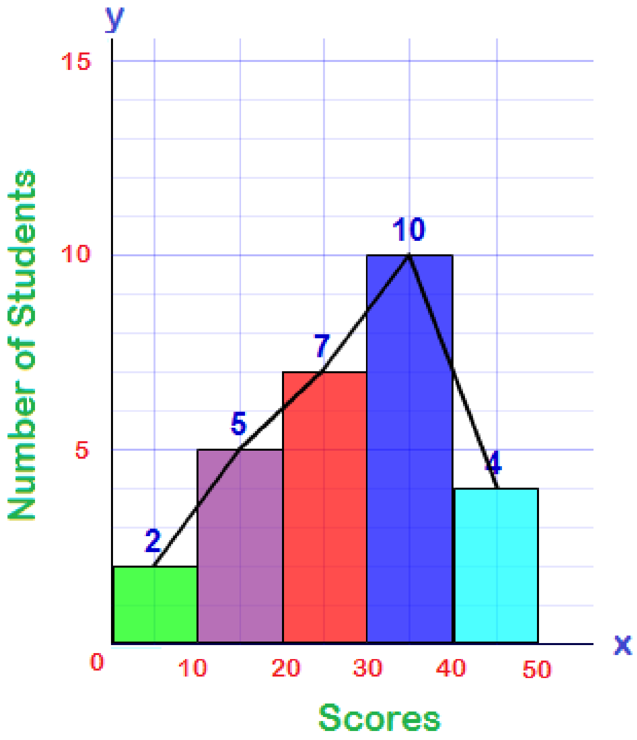
The histograms and frequency polygons are very common graphs in statistics. A histogram is defined as a graphical representation of the mutually exclusive events. A histogram is quite similar to the bar graph. Both are made up of rectangular bars. The difference is that there is no gap between any two bars in the histogram. The histogram is used to represent the continuous data.

A histogram may look like the following graph:



The frequency polygon is a type of graphical representation which gives us better understanding of the shape of given distribution. Frequency polygons serve almost the similar purpose as histograms do. But the frequency polygon is quite helpful for the purpose of comparing two or more sets of data. The frequency polygons are said to be the extension of the histogram. When the midpoints of tops of the rectangular bars are joined together, the frequency polygon is made. **Let us have a look at a sample of**

frequency polygon:



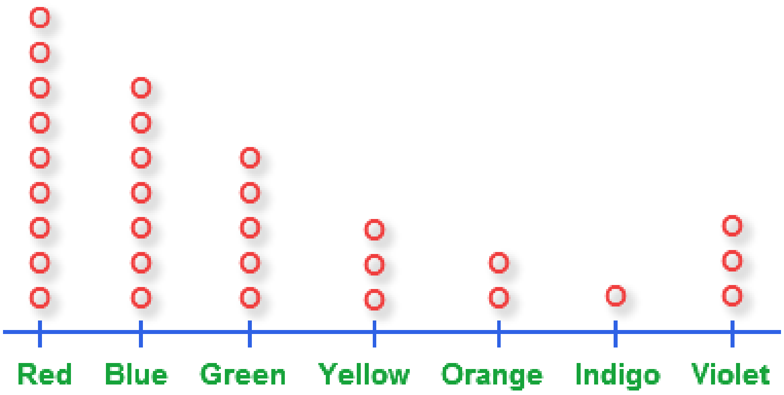
<u>Examples</u>

Few examples of graphical representation of statistical data are given below:

Example 1: Draw a dot plot for the following data.

Favorite Colors	**Red**	**Blue**	**Green**	**Yellow**	**Orange**	**Indigo**	**Violet**
Number of Students	9	7	5	3	2	1	3

Solution:
The line graph for the following data is given below:

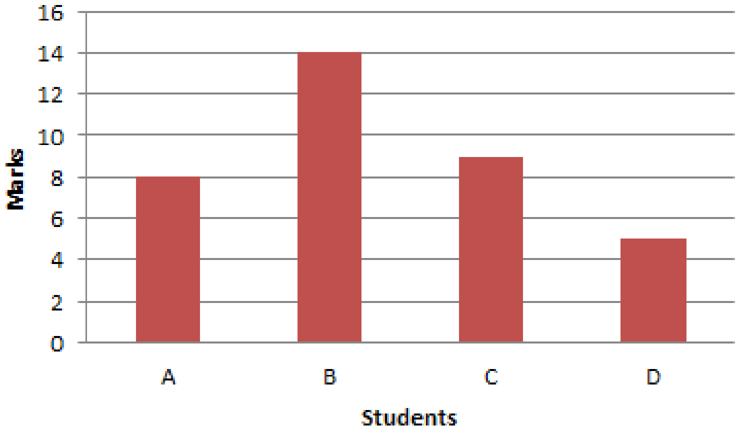


Example 2: Plot a bar graph from the data given below.

Students	**A**	**B**	**C**	**D**
Marks	8	14	9	5

Solution:

The following bar graph is obtained:

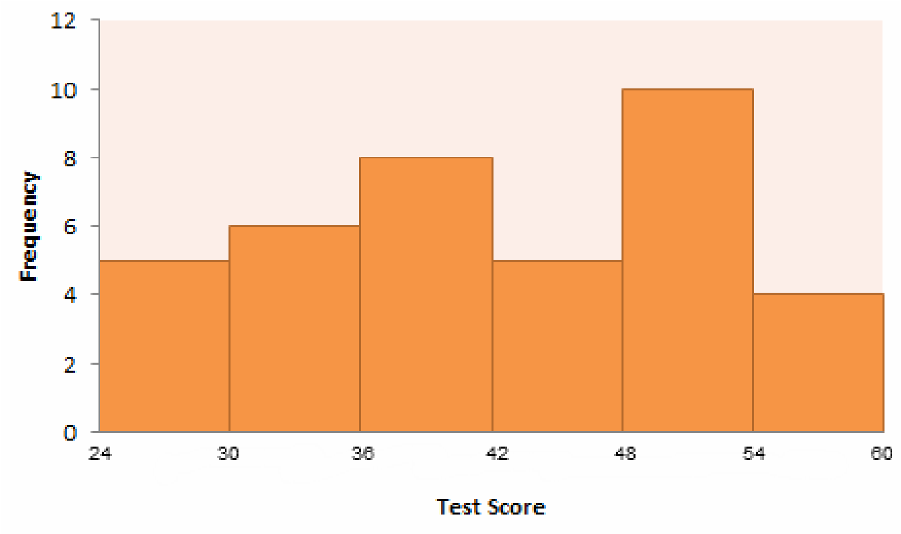


Example 3: Draw a histogram from the given data.

Test Score	**24-30**	**30-36**	**36-42**	**42-48**	**48-54**	**54-60**
Frequency	5	6	8	5	10	4

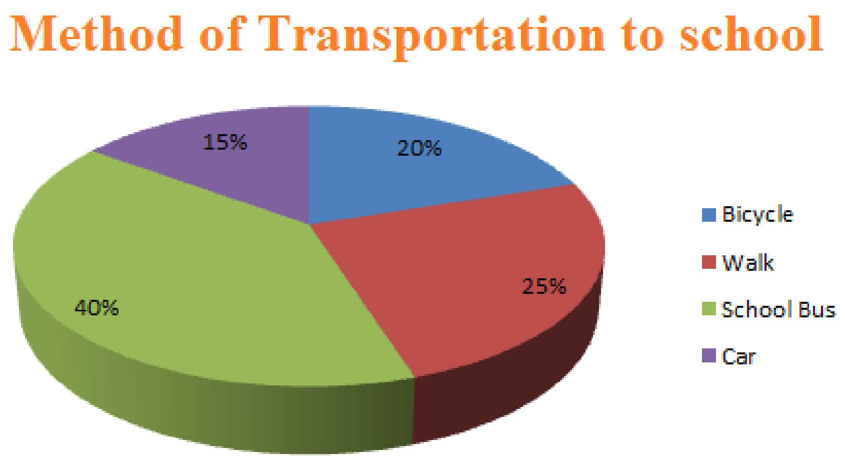
Solution:

We drew the following histogram:



Example 4: The percentages of students who use the different methods of transportation are as follows:
40% go by school bus
25% go by walk
20% go by bicycle
and rest 15% go by car. Draw a pie chart.

Solution: The pie graph of the above data is:



Spatial Model

Spatial modeling is an analytical process conducted in conjunction with a geographical information system (GIS) in order to describe basic processes and properties for a given set of spatial features.

The objective of spatial modeling is to be able to study and simulate spatial objects or phenomena that occur in the real world and facilitate problem solving and planning.

Spatial modeling is an essential process of spatial analysis. With the use of models or special rules and procedures for analyzing spatial data, it is used in conjunction with a GIS to properly analyze and visually lay out data for better understanding by human readers. Its visual nature helps researchers more quickly understand the data and reach conclusions that are difficult to formulate with simple numerical and textual data.

Spatial analysis or **spatial statistics** includes any of the formal techniques which study entities using their topological, geometric, or geographic properties. Spatial analysis includes a variety of techniques, many still in their early development, using different analytic approaches and applied in fields as diverse as astronomy, with its studies of the placement of galaxies in the cosmos, to chip fabrication engineering, with its use of "place and route" algorithms to build complex wiring structures. In a more restricted sense, spatial analysis is the technique applied to structures at the human scale, most notably in the analysis of geographic data.

Manipulation of information occurs in multiple steps, each representing a stage in a complex analysis procedure. Spatial modeling is object-oriented with coverage and concerned with how the physical world works or looks. The resulting model represents either a set of objects or real-world process.

For example, spatial modeling can be used to analyze the projected path of tornadoes by layering a map with different spatial data, like roads, houses, the path of the tornado and even its intensity at different points. This allows researchers to determine a tornado's real path of destruction. When juxtaposed with other models from tornadoes that have affected the area, this model can be used to show path correlations and geographical factors.

Symbolic Modelling

"We define Symbolic Modelling as a process, which uses Clean Language to facilitate people's discovery of how their metaphors express their way of being in the world."

Features of a Symbolic Model

- It contains a set of representations (or symbols) of something.
- It processes and manipulates those representations based on a set of rules programmed into the model.
- The rules operate on the representations according to their 'shape' or syntax, not according to what they represent (their semantics).

Let us take an example. The symbol '1' is character. In normal text, it represents the number one. When you read this text, you probably read it as a one. But this is a matter of interpretation, not a property of the symbol itself. For example, we could use the same symbol to represent the state of being 'on'. In fact, we do use it in this way on certain appliances — switches often have '0' and '1' marked on them to represent that the appliance is 'off' and 'on' respectively. The interpretation of the symbol (its semantics) is independent of the shape of the symbol (its syntax).

Here's another example: the string '11' represents, in normal English, the number eleven. But in binary, it represents the number three. In hexidecimal (a base 16 number system often used in different Internet protocols), it represents the number seventeen. None of these differences in interpretation affect the fact that it looks like two vertical strokes placed side by side. The rules that govern the processes of a symbolic model operate on how the symbol 'looks', not what it is interpreted to mean.

Symbolic Modelling proceeds through five defined stages, as follows:

- Stage 1: Entering the Symbolic Domain
- Stage 2: Developing Symbolic Perceptions
- Stage 3: Modelling Symbolic Patterns
- Stage 4: Encouraging Transformation
- Stage 5: Maturing the Evolved Landscape

Clean Language techniques are used throughout, to avoid contaminating or distorting the developing metaphor landscape through the form, content or presentation of the therapist's questions.